Natural deduction

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Why natural deduction?

- After all, we just found this nice method of truth-tables, which can be used to determine the validity or invalidity of any argument in truth-functional logic!

- But there are some problems with truth-tables

  - They can be very inefficient. The reason is that if you have an argument which contains k atomic formulas, then you'll need a truth-table with 2^k rows

  - The method of truth-tables doesn't work for logic as a whole. It is a very special feature of truth-functional logic that all its problems can be settled by a mechanical procedure.

- A proof is a demonstration that some statement is a logical consequence of others.

  - So a proof is an argument which is valid, and easily recognized as valid.

- Natural deduction is a system of construction of proofs.

- That is, when a statement is a logical consequence of others, we use natural deduction to demonstrate this.

- The system of natural deduction idealizes good human reasoning.

Simple and complex arguments

- There are some simple argument patterns which are clearly valid. For example, the patterns

  - "A, B, therefore $A \wedge B$".
  - "$A \rightarrow B$ and A, therefore B".

- Complex arguments can be built up from simple ones. For example
  - "$A \to B$, but also A. Therefore B. But, also C. Therefore $B \wedge C$."
- The reason this makes sense is. . .
  - *the result of correctly combining valid simple arguments is a valid argument.*

Natural deduction, the basic idea

- Think of an argument as a tree
  - the leaves of the tree are the assumptions
  - the root of the tree is the conclusion
- The tree "shows" that the conclusion is a logical consequence of the premises

A couple of rules of inference

- Begin with some simple argument patterns which are clearly valid, and represent them as simple trees. For example, in this way we get the patterns of
  - *conjunction introduction*:

$$\wedge\text{I} \, \frac{A \qquad B}{A \wedge B}$$

  - *modus ponens*:

$$\text{MP} \, \frac{A \to B \qquad A}{B}$$

- To say that these are valid patterns is to say that all arguments which follow these patterns is valid. So, e.g., both

$$\frac{p \wedge q \to r \wedge s \qquad p \wedge q}{r \wedge s}$$

  and

$$\frac{\neg(p \vee (q \wedge r)) \to (q \to r) \qquad \neg(p \vee (q \wedge r))}{q \to r}$$

  follow the pattern of modus ponens and are therefore valid.
  - So each simple rule already lets us recognize the validity of infinitely many arguments!
- We call a valid simple argument pattern "a rule of inference" when we've agreed to recognize it as valid

Combining steps

- By combining together simple trees which follow these recognized valid patterns, we obtain complex trees which also represent valid arguments. For example,

$$\cfrac{\cfrac{p \rightarrow q \qquad p}{q} \qquad r}{q \wedge r}$$

- This tree represents a valid argument.

    - the premises are $p \rightarrow q$, p, and r, and the conclusion is $q \wedge r$
    - the argument demonstrates that the conclusion is a logical consequence of the premises
    - recall that we write $AA \Rightarrow B$ to mean that B is a logical consequence of AA.

        * So the tree demonstrates that $p \rightarrow q, p, r \Rightarrow q \wedge r$.

- Here's a demonstration that $p \wedge q \rightarrow r, p, q \Rightarrow r$:

$$\cfrac{p \wedge q \rightarrow r \qquad \cfrac{p \qquad q}{p \wedge q}}{r}$$

The goal of completeness

- Now we can construct lots of trees!

    - We've so far recognized two simple argument patterns as logically valid
    - and we recognize correct combinations of valid arguments as valid too

- But, ultimately we want a system which is *complete* so that

    - whenever B is a logical consequence of AA, we can build a tree whose leaves are the formulas AA and whose root is the formula B

- So we need to add more rules of inference

A few more simple rules of inference

- The guideline for adding rules is just

    - a rule must be an obviously valid simple argument pattern

- The easiest case to finish is $(\wedge)$.

    - A formula $A \wedge B$ is true iff both A and B are true

– So we also get the pattern of *conjunction elimination*:

$$\wedge \text{E} \ \frac{A \wedge B}{A}, \frac{A \wedge B}{B}$$

- Using the two rules for conjunction we can prove some silly things like $A \wedge (B \wedge C) \implies (A \wedge B) \wedge C$

$$\frac{\dfrac{A \wedge (B \wedge C)}{A} \quad \dfrac{\dfrac{A \wedge (B \wedge C)}{B \wedge C}}{B}}{A \wedge B} \quad \dfrac{\dfrac{A \wedge (B \wedge C)}{B \wedge C}}{C}$$
$$(A \wedge B) \wedge C$$

- Of course, maybe this argument is so obviously valid that it didn't seem to need proof!

    – still, it's good that we can see some easy stuff, because validity isn't always obvious

- Another low-hanging fruit is *double negation*

    – recall that negation "flips" the truth-value
    – since there are only two truth-values, two flips takes us back to where we started
    – therefore we can endorse the *double-negation rules*

$$\text{DN} \ \frac{A}{\neg \neg A}, \frac{\neg \neg A}{A}$$

- In Chapter 3 we also recognized one other pattern, *modus tollens*:

$$\text{MT} \ \frac{A \to B \qquad \neg B}{\neg A}$$

- The double negation rule is often useful when combined with modus tollens:

    – For example consider this demonstration of $\neg p \to q, \neg q \Rightarrow p$:

$$\frac{\dfrac{\neg p \to q \qquad \neg q}{\neg \neg p}}{p}$$

- Part of the logic of disjunction is pretty easy

- recall that $A \vee B$ is true iff at least one of A and B is true

- so, you can use the rule of *disjunction introduction* ($\vee \text{I}$):

$$\vee \text{I} \ \frac{A}{A \vee B} \quad \frac{B}{A \vee B}$$

- So that's how you prove a disjunction

  - you might wonder, though: suppose we *have* a disjunction $A \vee B$ and want to prove something from it! if we only know that $A \vee B$ is true then we do know that at least one of A and B is true but don't know which one it is! We'll come back to that. . . .

Time for some exercises.

- At this point, try a few examples yourself!

  a. $p \wedge q \rightarrow r, p \rightarrow q, p \;\Rightarrow r$
  b. $p \rightarrow q, q \rightarrow r, \neg r \;\Rightarrow \neg p$
  c. $p \rightarrow (q \rightarrow r), p, \neg r \;\Rightarrow \neg q$
  d. $p \vee q \rightarrow r, p \;\Rightarrow r$

Natural deduction and validity

- Recall that a proof tree represents an argument
  - the leaves of the tree are the premises and the root of the tree is the conclusion
  - the proof tree demonstrates that the conclusion is a logical consequence of the premises
- also remember that we're in truth-functional logic!
  - so, the conclusion is a logical consequence of the premises provided that
    * there is *no row of a truth-table on which the premises are true and the conclusion is false*
- You build proof trees from smaller ones
  - If you want to make sure you're doing it right:
    * take note that at every step of construction, the formula at the root of your tree remains a logical consequence of the formulas at the leaves!
    * you can use a truth-table to verify this

Conditional proof

- Let's now get to the heart of natural deduction
- Note that people often argue like this:

- – "(1) Suppose that Tony talked. Well (2) if Tony talked then Lisa laughed. So (3) Lisa must've laughed. But (4) if Lisa laughed then Fred frowned. So, (5) Fred frowned. But (6) this means that *if* Tony talked, then Fred frowned."
- – Using the resources developed so far we can diagram *almost* all of this argument:

$$\cfrac{\cfrac{(1)\ p \qquad (2)\ p \to q}{(3)\ q} \qquad (4)\ q \to r}{(5)\ r}$$

- • This diagram demonstrates that $p, p \to q, q \to r \Rightarrow r$.
- • However, what the English argument demonstrates is $p \to q, q \to r \Rightarrow p \to r$
- • So to complete the diagram we must take a further step:

$$\cfrac{\cfrac{(1)\ p \qquad (2)\ p \to q}{(3)\ q} \qquad (4)\ q \to r}{(5)\ r} \quad \rightsquigarrow \quad \cfrac{\cfrac{\cfrac{(1)\ \boxed{\times}\ p \qquad (2)\ p \to q}{(3)\ q} \qquad (4)\ q \to r}{(5)\ r}}{(6)\ \boxed{p \to r}}$$

- • The new argument demonstrates that $p \to q, q \to r \Rightarrow p \to r$
- • In the last step two things happened:

  i. we weakened the conclusion from r to $p \to r$
  ii. we *discharged* the assumption of p by marking it with $\times$

- • The new conclusion is a weaker conditional statement

  - – The new conclusion *no longer depends on the discharged assumption*
  - – Thus, a leaf marked with $\times$ is not really a leaf (pretend it fell off)

Another example of conditional proof

- • Before going on, let's consider another example: $p \to r \Rightarrow p \land q \to r$

  - – you might wonder, is this really valid?
  - – if so, then this can be checked! try doing a truth-table. Or you can intuitively explain it to yourself like this. "Suppose that the conclusion is false. Then $p \land q$ must be true while r is false. But if $p \land q$ is true, then p is true. Hence p is true while r is false. But that means that the premise is false! So the premise must be false whenever the conclusion is false. I.e., yes it's valid."

    * review note: use truth-table to show conversely that $p \land q \to r \not\Rightarrow p \to r$

- • Now to the proof

- – our aim is to prove a conditional
- – to do this we take three steps:
    1. assume the antecedent
    2. deduce the consequent
    3. infer the conditional and discharge the assumption of the antecedent

- step 1 is easy: assume, for the sake of argument, the antecedent of your desired conditional:

$$p \land q$$

- Now for step 2, try to prove the consequent (using the other premises as necessary):

$$\frac{\dfrac{p \land q}{p} \quad p \to r}{r}$$

- At this point, we have deduced the consequent of our desired conditional, using the premise $p \to r$ and the temporary assumption $p \land q$

    - – so we've now shown that the r is a logical consequence of $p \to r$ and $p \land q$

- Time for step 3:

$$\frac{\dfrac{\dfrac{\overset{\times}{p \land q}}{p} \quad p \to r}{r}}{p \land q \to r}$$

and that's it.

- As before, step 3 has two parts:

    i. weakening the conclusion from r to $p \land q \to r$
    ii. discharging the assumption of $p \land q$

- These two parts are complementary:

    - – it is easier for $p \land q \to r$ to be true than it is for r to be true, but
    - – it is harder to deduce something from $p \to q$ than it is to deduce something from $p \to q$, $p \land q$

Conditional proof, the general idea

- The rule of *conditional proof* is a bit different than the earlier ones

    - – we don't just change the conclusion, but also "snip" a leaf from the tree!

- More generally it looks like this:

$$
\begin{array}{c}
\boxed{
\begin{array}{cc}
 & \times \\
A & A \\
\vdots \quad \rightsquigarrow & \vdots \\
B & B \\
 \text{CP} & \overline{A \to B}
\end{array}
}
\end{array}
$$

- Thus, conditional proof transforms a proof of $A, \cdots \Rightarrow B$ into a proof of $\cdots \Rightarrow A \to B$.

Nested conditional proofs

- to prove a conditional, you begin by assuming its antecedent "for the sake of argument"

  - then you go on to deduce the consequent

- But what if you want to prove a conditional whose consequent is also a conditional?

  - Naturally, assume its antecedent too!

- For example consider $p \wedge q \to r \Rightarrow p \to (q \to r)$.

  - You want to prove $p \to (q \to r)$. So with an eye to deducing its consequent, begin by assuming its antecedent:

$$p$$

  - Now you want to prove $q \to r$. So with an eye to deducing its consequent, assume *its* antecedent!

$$p \qquad q$$

  - Now you just want to prove r. You can't break that down any more. So take a peek at the original premise $p \wedge q \to r$, and start bashing away:

$$
\dfrac{\dfrac{p \qquad q}{p \wedge q} \qquad p \wedge q \to r}{r}
$$

  - Look, now you've got r!

  - But remember, we only wanted to prove r on the basis of $p, q, \cdots$ so as to show that $q \to r$ follows from $p, \cdots$, thus:

$$
\dfrac{\dfrac{\dfrac{p \qquad \overset{\times}{q}}{p \wedge q} \qquad p \wedge q \to r}{r}}{q \to r}
$$

– And we wanted to prove $q \to r$ on the basis of $p, \cdots$ so as to show that after all $p \to (q \to r)$ follows from $\cdots$, thus:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{✌}}{p} \quad \overset{\text{☺}}{q}}{p \wedge q} \qquad p \wedge q \to r}{r}}{\text{☺} \;\dfrac{}{q \to r}}}{\text{✌} \;\dfrac{}{p \to (q \to r)}}$$

– When you're discharging one assumption inside another, it's a good idea to indicate which you've discharged where. You can use whatever symbols you want for this.

Conditional proof and validity

- At this point you might wonder...

  "yeah, I could see how the other rules were valid from the truth-tables, but this one is pretty weird! what's the deal?"

  – in other words, you may not be persuaded that conditional proof preserves validity

- So here is a little argument to persuade you skeptics. (If you're not a skeptic, just trust me and skip it.)

  – Note that whenever we do conditional proof, we transform a proof of $A, CC \Rightarrow B$ into a proof of $CC \Rightarrow A \to B$
  – Could this have turned a valid argument into an invalid one?
    * Well, let's suppose that $CC \Rightarrow A \to B$ is invalid
    * Then, there must be a world W at which CC are all true and $A \to B$ is false
    * So at W, we must have that A is true and B is false
    * But then at W, we have that A, CC are all true while B is false
    * So $A, CC \Rightarrow B$ must've been invalid too!
  – So actually, *every result of appling CP to a valid argument is a valid argument*
  – That is to say, the rule of conditional proof is deductively sound

More exercises.

- Again, now's a good time to try some yourself! (You might start by trying to rebuild on your own the ones presented above.)

  a. $p \to \neg q, r \to q \Rightarrow p \to \neg r$
  b. $p \to (q \to r) \Rightarrow p \wedge q \to r$

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c. $p \to (q \to r) \Rightarrow q \to (p \to r)$
d. $p \lor q \to r \Rightarrow p \to r$
e. $p \to r, (p \land q \to r) \to s \Rightarrow s$