Logic in the *Tractatus* II

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Contents

1 Is logic a science?

Frege begins the 1879 *Begriffsschrift* by distinguishing the way that a truth happens to become known, from the best possible foundation the truth could receive. How people happened to discover something may have little to do with what makes it so. But, following the accident of its discovery, people begin to seek out the grounds upon which a truth rests. The kind of grounds to which this deepening grasp will lead, depends on the truth's particular nature.

According to Frege, the most reliable kind of demonstration of a scientific truth would disregard all particular characteristics of objects, and depend only on laws on which all knowledge rests. As is well known, Frege strove to uncover such a source for the truths of number. This would yield a rigorous proof of the laws of number on the sole basis of maximally general laws of logic. The value of such a proof would not consist in its raising human subjective confidence in the truth of the laws of arithmetic. Rather, it would reveal their place in a natural justificatory order. Basic logical laws would appear at the root of a science of pure reason, which develops upward through the expression of further logical truths and blossoms into the science of number. Thus, in devising his ideography, Frege had in mind "right from the start", the "expression of a content" (Frege, 1979, 12): the contents of general logical laws, of arithmetic theorems, and finally of the application of all these rational truths throughout the special sciences of astronomy, chemistry, biology and so on.

So being a rational science, logic ideally unfolds through judgments which stand in a natural order of justification.[1] A given truth of logic is shown to rest on other truths, and those on others, and at the bottom there are basic laws of logic, the truths which are basic to the science. Of course, the logician does not simply tell us that logical truths follow from truths which are basic. Rather, the logician makes this following manifest, by reducing the queried instance of consequence to a series of applications of antecedently recognized, fundamental principles of thought. Frege spent years building proofs in the way so envisioned. But Frege's arguably deeper achievement was to formalize his general notion of proof at all. One might say that in drawing attention to those fundamental principles which suffice for the demonstration of truths of logic, Frege aimed to give the general form of the theorem.

In the 1879 passage I quoted above, what's translated as "proposition" is *Satz*. Frege speaks of "how we have arrived at a proposition" as the manner of an instance of apprehension of truth.[2] More generally, here Frege follows German mathematical tradition in using *Satz* to mean something like "statement (to be) proven".[3] So the preface concludes with a promise that Frege's ideography may supply a deeper foundation for its theorems (*Sätze*).[4] And throughout the body of the work, Frege generally refers to the numbered proven results as propositions. In contrast, for the sort of thing which can be judged (unlike a concept of a house), he uses not "proposition" but "judgeable content".[5] On this usage in the *Begriffsschrift*, paradigms of propositions are

[1] [[Blanchette]]

[2] *Es kann daher einerseiz nach dem Wege auf dem ein Satz allmählich errungen wurde, andrerseitz nach der Weise, wie er nun Schliesslich am festesten zu begründen ist*.

[3] Cf. Martin-Löf (1996).

[4] (Frege, 1879, XIV).

[5] There are a couple of exceptions, representing another strand of usage of *Satz*. For example, at the end of §4 Frege addresses the Kantian classification of the modality of judgements. Insodoing, he considers what people are doing when they represent a *Satz* merely as possible. Here, the sense of *Satz* would instead be something like judgeable content. Later in his career, in the more language-oriented context of, for example, Frege (1892), of course *Satz* usually just means "sentence" [[ref]].

recognized truths of logic.

To this Wittgenstein responds, poignantly: there is no such thing as a logically true proposition. Far from being redundant, "proposition logically true" is a near-contradiction in terms. The point of a proposition is to say how the world is, in contrast to some other ways in which the world might be. But if the truth of a proposition were required unconditionally, then it would draw no contrast. For Wittgenstein, a proposition is essentially not logically true. It is, essentially, logically possibly true and logically possibly false. Propositions express possibilities but only mere possibilities. A proposition which is logically true is therefore in some sense a degenerate, or limiting case.

But logic is supposed to provide a standard of knowledge. And it must be wondered whether the concept of possibility could bear such weight—surely, Frege would deny this, as would Wittgenstein's own teacher Russell. For one thing, the concept seems just unclear. Is there is a possibility that it have been raining as you read this? Or that it be 2:35 in Vienna? Or 2:35 on the sun? How about that $2+3 \neq 5$? In the *Tractatus*, Wittgenstein tries to meet this challenge. In general, he supposes that possibility may be spelled out in terms of others, and those through yet others, and so on. But then again, there are possibilities Wittgenstein takes to be, let's say, basic: their possibility is not to be explained in turn (just as the truth of a Fregean axiom is not to be explained by reduction to other truths). In this way, non-basic possibilities become clarified as positions with respect to the obtaining and non-obtaining of the basic possibilities. Whether or not a possibility obtains is to be spelled out in terms of which basic possibilities obtain. Indeed, not only is the obtaining of a possibility equivalent to the realization of some pattern of obtaining or nonobtaining of the basic possibilities; rather any possibility is itself nothing more than such a pattern.

For Wittgenstein, the supervenience of possibilities on basic possibilities gets made evident through analysis. Analysis reduces an instance of such supervenience of a possibility to the repeated application of truth-operations. Analysis thus represents a possibility as the result of repeatedly applying joint affirmation, denial, etc., to expressions of possibilities which are basic. What Wittgenstein himself does in the *Tractatus* is to codify the notion of analysis, by giving what he calls the general form of the truth-function. So, whereas Frege codifies the general form of the logically true and says, "go find some proofs", Wittgenstein codifies the general form of the logically possible, and says "go find some analyses".

The upshot of analysis is quite different from the upshot of proof. For the upshot of proof is scientific knowledge, i.e., recognition of the truth of the theorem proven. The upshot of analysis is no more than clarity. This deflating of intellectual ambition seems characteristic of the attitude expressed in Wittgenstein's remark, in the preface of the *Tractatus*, that one of the achievements of that book might be to have shown how little is accomplished when philosophical problems are solved.

So here is the basic idea: Wittgenstein's approach to logic replaces Frege's codification of proof with a codification of analysis. In this way, recognition of logical truths gives way to clarification of logical possibilities. So broadly speaking, Wittgenstein's is a transformation of Frege's; see Figure 1.

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    Logical --------justification---------> Basic truths
    truths             |
                       |
                       |
                     Proof--------codification------> General form of
                       |                              the theorem
                       |
                       |
                       V
                   Inference


   Logical --------determination--------> Basic
Possibilities          |              possibilities
                       |
                       |
                   Analysis------codification--------> General form of
                       |                               the truth-function
                       |
                       |
                       V
                Truth-operations
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Figure 1: Wittgenstein's transformation of Frege's program

1.1 Two challenges for Wittgenstein's response

That's a fairly natural, if schematic story about how Wittgenstein responds to logicism. But it confronts a couple of immediate challenges. Consider the core disagreement about the status of logical truths. In maintaining, against Kant, that logic yields genuine knowledge, Frege actually had an argument. For example, the third part of the *Begriffsschrift* presents an analysis of the ancestral of a relation. That analysis underwrites the proof of theorems along the lines of "a parent of an ancestor is an ancestor of a parent". It's not without justification, then, that Frege exults that thought by itself can produce judgments which might've seemed possible only on the basis of intuition.[6]

Against this, Wittgenstein baldly declares that propositions of logic are "tautologies" (6.1). With characteristic generosity he explains: "the propositions of logic therefore say nothing", and "all theories that make a proposition of logic appear to have content are false" (6.11,6.111). What does Wittgenstein mean by this alleged emptiness of logical truths, and what is its justification?

One interpretation is that Wittgenstein uses "tautology" as a slur: perhaps a tautology is a proposition which doesn't merit scientific interest because it's logically true. For example, in ordinary German, the term *Tautologie* meant what it still does, something like "redundancy", or "pleonasm". Moreover, as Dreben and Floyd 1991

[6]Frege 1879, translation from van Heijenoort 1967 (55)

observe, Kant had designated as tautological those judgments which are analytic on their face, like for example "unmarried men aren't married". So, Wittgenstein is reflecting back at Frege what was Kant's reply to Leibniz. Unfortunately, Kant's reply had force because the analyticities he described really do seem vacuous. And to this, we just saw what Frege has ready: check out Part III of my book! So it is no good for Wittgenstein simply to mimic the Kantian putdown.

Maybe there was more to "tautology" than mere derogation. At 4.46, Wittgenstein had already reintroduced the word as a term of art. I'll return to the details later. But, let's suppose he stipulated for "tautology" something like its now-familiar technical meaning: a tautology is a formula which is true in virtue of its truth-functional structure. The remark that logical truths are tautologies would surely be inflected such a stipulation.

But so understood, it still seems to miss the point. It is literally false of Frege's system its theorems are true in virtue of truth-functional structure. And anyway, mere stipulation can't defuse the appearance that Frege's theorems are interesting! So a first challenge for Wittgenstein's attempted transmutation of logic is this: can Wittgenstein's account of the nature of logical truth be extended to a more expressive system? And can that extension sustain the viewpoint that logical truths are in some sense degenerate or vacuous?

The second challenge concerns a consequent tension in Wittgenstein's understanding of his teachers. Wittgenstein certainly learned some logic from Frege and Russell,[7] and he knew this. Even in 1912, he wrote that Frege and Russell "brought about an advance in Logic comparable only to that which made Astronomy out of Astrology, and Chemistry out of Alchemy" **?**. But suppose he's right that logical truths are vacuous. Then, it would have to seem as though Frege and Russell devoted years of their lives to the senseless transformation of empty formulas into empty formulas by emptiness-preserving rules—this in the face of Frege's emphatic insistence that, "right from the start" he had as his aim with ideography the "expression of a content."[8] So the second challenge is this. How could Wittgenstein reconcile the logical achievements of Frege and Russell with what he takes to be their profound misconception of logical truth and of logic more generally? From a *Tractatus* point of view, what is the value for logic of Fregean proof?

In the next subsection, I'll begin by addressing the second challenge, about how Wittgenstein understood his teachers. This will point the way to a partial solution to the first challenge: but determining how logical truth could be understood as empty from a *Tractatus* viewpoint will require the rest of the paper.

1.2 The logical significance of Fregean proof

It is in the 6.12s that Wittgenstein acknowledges the established interest and power—such as he found it—of logic in the hand of the logicists:

> We prove a logical proposition by creating it out of other logical propositions by applying in succession certain operations, which again generate

[7] Bolzano.

[8] This irony is noted by Juliet Floyd (Floyd and Shieh 2001, 154).

> tautologies out of the first. (And from a tautology only tautologies *follow*.) [6.126c][9]

This passage illustrates Wittgenstein's ambivalence toward the logical achievements of Frege and Russell. On the one hand, for example, at 3.325 Wittgenstein urges upon philosophers the use of a "sign-language that is governed by *logical grammar*—by logical syntax." And, he says, "the conceptual notation of Frege and Russell is such a language, though, it is true, it fails to exclude all mistakes." Wittgenstein finds that Frege and Russell correctly aim at constructing a symbolism which does not just ape the whims of words but respects the forms of logical syntax—even if, in pursuit of this aim, they sometimes fall short. So throughout the *Tractatus* Wittgenstein fusses over details of implementation within the logicist machinery.[10] This preoccupation in itself indicates that there's something within the unfolding of logicism which he does find illuminating. But what?

Wittgenstein acknowledged that Frege's discovery of the quantifier reveals something of the essence of propositions. At 4.0411, he considers various alternative notations for generality. For example, perhaps the universal generalization of a formula Fx could be expressed simply prefixing the letter G to it. The point he wants to make is that the alternatives don't work. One cannot, in the expression of generality, "get away" from the kind of representational structure that the quantifier notation makes explicit.[11] Thus, the quantifier notation embodies an important insight. More generally, the Frege-Russell presentation of logical structure in a symbolism must be sufficiently close to the mark that the fact that some formation of signs yields a tautology does tell us something. Still, what's that?

There must be something—if not to how things must be if a tautology is true—at least, to the fact that a proposition is a tautology, which explains how such facts could have so entranced the inventors of serious logic.

> The fact that the propositions of logic are tautologies *shows* the formal—logical—properties of language, of the world.
> That its constituent parts connected together *in this way* characterizes the logic of its constituent parts.
>
> In order that propositions connected together in a definite way may give a tautology they must have definite properties of structure. That they give a tautology when *so* connected shows therefore that they possess these properties of structure. [6.12]
>
> The propositions of logic demonstrate the logical properties of propositions, by combining them into propositions which say nothing.
> This method could be called a zero-method. In a logical proposition propositions are brought into equilibrium with one another, and the state

[9] Unless otherwise marked, translations of the *Tractatus* in this paper taken from Ogden-Ramsey 1922.

[10] For example, he complains that Frege and Russell "introduce generality in association with logical product or sum", which "made it difficult to understand the propositions in which both ideas are embedded" (5.521). And, "Russell's definition of '=' is inadequate" (5.5302). And again, Frege's account of the ancestral "contains a vicious circle" (4.1273).

[11] In this paragraph, I'm indebted to Kremer (1992).

> of the equilibrium then shows how these propositions must be logically constructed. [6.121]

These passages present Wittgenstein's error-theoretic interpretation of logicism. Since Wittgenstein holds that logical truths are not truths at all, there cannot be such a thing as a science whose proper truths are the propositions of logic. Nonetheless, the notion identified by Frege, that of being a proposition of logic, must have some kind of logical significance which explains how it could have so entranced him. In the passages just quoted, Wittgenstein sketches a manner of inquiry in which interest in the property of being a tautology would take the slightly contorted form it deserves. This form of interest is the following: that a tautology results from combining some propositions in a certain way discloses something about the structure of those propositions. Wittgenstein gives examples of this sort of observation not just in the not 6.12s, but also in the 4.12s and the 5.13s. That conditionalizing p on q yields a tautology tells us that q entails p; that a tautology results by negating the conjunction of p with q tells us that p contradicts q.[12]

So, in the 6.12s, Wittgenstein sketches out a distinctive distribution of knowns and unknowns that characterizes a certain kind of inquiry. What we don't know, and what we want to investigate, are the internal structures of some propositions, say p and q. On the other hand, we know how to combine p and q into a single proposition (5.131) whose sense is a function of the senses of p and q (5.2341). And, moreover, since a tautology must itself show that it is a tautology (6.127), we can tell whether or not a tautology results by combining p and q into a single proposition. Consider a simple case, in which the analyst combines queried propositions p and q into the assemblage $\neg(p \wedge q)$, where, it turns out, this is a tautology. The analyst knows that negation of $p \wedge q$ is true if and only if $p \wedge q$ is false, that $p \wedge q$ is true if and only if p and q are both true, and furthermore, that a tautology is always, in a degenerate sense, true (4.46ff). In this way, that $\neg(p \wedge q)$ is a tautology shows the analyst that p and q are never both true, or in other words, that p contradicts q. So, for example, the analyst might record that "there are at most three chicks in the nest" contradicts "there are at least five chicks in the nest". Systematic consideration of entailments like this might eventually lead to a quantificational analysis of judgments of number of the sort that Frege rejects in the *Grundlagen*.

Still, this method of inquiry looks a little contorted. Why is it necessary at all to combine the queried propositions into a single one and ask whether it is a tautology? Can't we already discern between the queried propositions themselves some logical relationship which guarantees the tautologousness of their combination? That is, isn't it enough just to consider the relations between the constituent sentences "there are at most three chicks", "…at least five", etc.? Wittgenstein puts the point as follows.

> If the truth of one proposition follows from the truth of others, this expresses itself in relations in which the forms of these propositions stand to one another, and we do not need to put them in these relations first by connected them with one another in a proposition; for these relations are

[12] 4.1211; 5.13-5.131; 6.1221.

internal, and exist as soon as, and by the very fact that, the propositions exist. (5.131)

So, the circumstance that one proposition follows from some others does not depend on the logical truth of the corresponding conditional. Nor is it tenable to suppose that the knowledge of logical truth of corresponding conditionals is always prior to the knowledge of consequence: rather, the knowledge of consequence may receive self-standing realization in inference (5.132).

The seemingly cryptic 5.131 becomes rather straightforward, when seen as part of an error theory, or psychological explanation of how deep insights into propositional structure could be nurtured within a misguided conception of logic. As logicians, Frege and Russell can't avoid wanting to understand what makes a conclusion follow even from false premises. But since logical consequence finds no direct expression within their systems, they sublimate their curiosity into what, for Wittgenstein, is a sterile enumeration of tautologous conditionals. In turn, this process explains the appearance that logic has a subject-matter. The reflection of internal relations between propositions in the tautologousness of logical combinations of those propositions gets mistaken, by the logicist, for interesting facts which tautologies report. Russell's habitual gloss of $p \supset q$ as "p implies q" would definitely look like a Freudian slip. In this way, the preoccupation with logical truth cooperates with a confusion of material relations with formal ones, a confusion which is "very widespread among philosophers" (4.122).

As I see it, then, 5.131 forms part of an error theory to explain how misconceived developments of logic could nourish deep insights. In contrast, Proops (2002, 288) describes 5.131 as "darkly metaphorical", alluding severally to explicit analyses of the consequence relation in Frege and Russell. In particular, Proops extracts a proof-theoretic account of logical consequence from some darker corners of the Fregean opus.[13] Thus, Proops finds Wittgenstein squarely to attack the positive accounts (such as they are) in Frege and Russell of the consequence relation. One difficulty for this reading is that as Goldfarb (2001) and Proops himself (1997) did well to show, Wittgenstein was no Frege scholar. Moreover, Frege is consciously dismissive of the pretheoretical concept of entailment in the first place, perhaps even doubting its cogency. So it is not clear why he should be much bothered by criticism of his account of it—so much the worse for your concept, he might reply. As straightforward philosophical criticism, 5.131-2 is not very compelling: it invites the reply that nobody expects confused or misunderstood notions to have straightforwardly satisfying analyses. On the other hand, somebody's overt and sincere disavowal of interest in X doesn't imply that interest in X doesn't explain their behavior. Wittgenstein was Viennese, after all. So it's more natural to suppose that in 5.131 Wittgenstein operates diagnostically: the assertion of tautologous conditionals is a deformed expression of the natural logical interest in logical consequence.

A second point of contrast with Proops is this. Proops finds the 5.13s to be specifically aimed at the notion of logical consequence. But on the reading advocated here, the drift of the 5.13s is continuous with the 6.12s and diagnostic of obsession with all kinds of tautology, including biconditionals, negations, etc., and correspondingly

[13] Mainly (1906, 333ff), but also *Grundlagen* (1884, §17).

finding the actually interesting underlying facts to be not just entailments but also equivalences, incompatibilities, etc. So, on the present reading, it is only because Frege asserts so many conditionals, i.e., only because his system contains a primitive sign for conditional rather than for disjunction that Wittgenstein focuses on logical consequence in the 5.13s.

1.3 Reviving "tautology"

Let's return to the main thread. Recall that for the simple story of §1.1, there was another challenge. Behind Wittgenstein's assertion that logical truths are tautologies, is there any more than flatfooted insistence? This won't stand its own ground, certainly not before his teachers' hard-won virtuosity. As I mentioned earlier, his use of "tautology" invokes both traditional putdown of logical truth and also his own technical stipulation. Let me now sketch part of what revives the traditional putdown.

The logicists conceive of logic as a science in its own right, a naturally ordered system of truths. According to Wittgenstein, it's this conception of logic as a science which obscures the range of phenomena which really interest the logician—like, for example:

> [...] One can *draw inferences* from a false proposition. [4.023e]

This remark has the tone of an exasperated reminder, of a familiar fact that's submerged by the logicist conception. As is well known, Frege denied it explicitly [[]]. But I'm urging that Wittgenstein didn't need to know this, because the denial is essentially obvious from the way in which logic unfolds in *Begriffsschrift*: to infer is to judge on the basis of other judgments.

More generally, Wittgenstein acknowledges that within logicist activity, natural logical purposes do find some expression. Systematically asserting only logical truths is at least a rather intimate flirtation with what's really interesting: that this-or-that proposition is logically true. In turn, the system of assertions becomes a procrustean bed for a broader range of logical phenomena: primarily not features of individual truths, but also relations between propositions or even features of whole multiplicities of propositions.

Even a procrustean bed doesn't fit everything. Logically, people care just as much about what doesn't follow from the premises, about which hypotheses can be true simultaneously, and so on. These are questions not about what's true, but about what's possible. And from within the logicist point of view, questions of possibility have at best various highly inequivalent approximations. Frege's (1879) discussion is representative. First, Frege proposes that in saying that a proposition is possible, a speaker may mean to indicate that she suspends judgment about it because she cannot refute it from general laws. But Frege and Wittgenstein agree that such cognitive relations of a speaker to a proposition are logically meaningless.[14] Frege also proposes that in calling a proposition possible a speaker may mean that its universal generalization is

[14] For example, at 4.442 Wittgenstein rejects the use in Frege and Russell of the turnstile, on the grounds that it serves only to indicate such a relation.

false. Fairly enough, Wittgenstein responds at 6.1231: "to be general means no more than to be accidentally valid for all things."[15]

So, what partly revives the traditional putdown is this: logical truth doesn't merit the scientific interest it's accrued. By trying to squeeze the variety of logical phenomena into the single notion of logical truth, logicism can't leave a fundamental place in logic for the concept of possibility.

> ([…] Logic deals with every possibility and all possibilities are its facts.) (2.0121c)

As logicians, then, Frege and Russell seemed to the earliest Wittgenstein like heroes who made astronomy from astrology or chemistry from alchemy. Yet he came to feel that logicism frustrates what are natural logical interests. But what are the natural logical interests, anyway?

2 The application of logic

According to Frege and Russell, logic has nothing specifically to do with people. But conversely, logicism explains what people have to do with logic. Logic is a general science, whose basic facts everybody should know who wants to reason correctly about plants and animals, raindrops and polyhedra. We just saw that Wittgenstein rejects the idea that logic is a science. So, instead, what are people supposed to do with it?

In this section, I'll argue that central features of the *Tractatus* can be understood as an attempt to answer the question what people have to do with logic. In short, logical activity is analysis. Instead of truths reduced by proof to axioms, possibilities are reduced by analysis to patterns of agreement and disagreement with basic possibilities. This, as we'll see, is the application of logic, which shows what propositions must have in common with the world in order that they be propositions at all.

This conception of the role of logic in human life constrains how propositions can be understood to participate in logical relationships. In this section, I will try to spell out those constraints. Specifically, I will argue that so understood, logical relationships must supervene on what makes anything into propositional symbols: in short, logical relationships must be symbolically realized. I will furthermore argue that a range of received interpretations of the *Tractatus* fail to respect this characterization. This will set the stage for my attempt, in §§3 and 4, to construct a better account.

I'll begin in §2.1 with a capsule summary of Wittgenstein's conception of analysis, aiming to highlight one of its well-known commitments: that representation and reality share a common form. In §2.2, I'll point out some central features of Wittgenstein's inheritance from Frege and Russell which make it hard to explain how analysis could come to be needed at all. In §2.3, I'll argue the sign-symbol distinction opens some room for Wittgenstein to explain how propositional identity could become obscure. In §2.4, I'll explain just what it is that analysis is supposed to clarify: the manner in which propositions "mirror reality". In §2.5, I'll combine the sign-symbol distinction and

[15] In response to Russell's similar proposal that a propositional function is possible when its existential generalization is true, Wittgenstein declares at 5.525 that this analysis is "incorrect".

the analogy of propositions with places to explain the point of analysis. Finally in §2.6, I'll argue that this conception of analysis constrains in turn how logical relationships could be constituted: logical relationships must supervene on what makes anything into propositional symbols.

2.1 Understanding dependence

Let's start with an example.

(p) The surface reflects white light.

(p_0) The surface reflects red light.

(p_1) The surface reflects orange light.

⋮

(p_6) The surface reflects purple light.

For concreteness, let's suppose that for the possibility presented by p to obtain just is for all of the possibilities presented by the p_i to obtain.

That something's reflecting white light is *ipso facto* its reflecting red light would seem to be an empirical discovery. Then it would seem natural to record the discovery as $p \to p_0$. But according to Wittgenstein, this is a mistake. For what p reports to depend on what the p_i report in the way we've supposed, is precisely to stipulate, among other things, that p affirms each of the p_i. Thus, $p \to p_0$ doesn't disagree with any possibility; it doesn't narrow down the location of the actual world in logical space. Indeed, each of the $p \to p_i$ say the same thing, which again is the same as what $p \to p$ says.

So the attempt to report functional dependencies is a bit like the logicist's enumeration of tautologies. The situations presented by p and the p_i bear to each other relations that are, in the jargon, "internal". And,

> The existence of an internal relation between possible [situations (*Sachlagen*)] expresses itself in language by means of an internal relation between the propositions presenting them. [4.125]

Of course, it is some kind of improvement in understanding for people to discover that reflecting white light just is, among other things, to reflect red light. What constitutes the improvement if not discovery of a fact? Wittgenstein's proposal is this:

> We can bring out these internal relations in our manner of expression, by presenting a proposition as the result of an operation which produces it from other propositions [...]. [5.21]
>
> Denial, logical addition, logical multiplication, etc., etc., are operations. [5.2341b]

For example, the internal relation which p bears to the p_i can be "brought to prominence" by presenting p as the result of an operation on the p_i, thus:

$(p') \ \bigwedge(p_0, p_1, \ldots, p_6)$

Since p is just that which is true iff each of the p_i is true, therefore the coordinating mark $\bigwedge$ makes it possible to use the p_i to form an expression of p. More generally: internal relations are apprehended not by judgment, but by a shift in the means of expression. Such a rewriting makes evident something of what an antecedently understood proposition says.

Let me summarize how this conception of the use of logic for people can so far be seen to differ from the use of logic envisioned in logicism. For the logicists, logic unfolds in a self-standing body of logical truths, which get applied in the other sciences by instantiation. Logic becomes known to people through acknowledgement of its truths in judgment. Wittgenstein rejects this conception. Rather, logic comprises those internal relations of entailment and exclusion in virtue of which each proposition, true or false, finds its logical place in the system of all propositions. As for what people have to do with this: logic can't become known through acknowledgement of its truths, because it doesn't bring its own truths to acknowledge. Human logical activity is directed toward the application of logic. To apply logic is to rewrite propositions already understood, so that internal relations become apparent.

2.2 Picturing as apprehension

As it's just been characterized, analysis investigates the interdependence of possibilities. Much of that interdependence could be uncovered only by empirical inquiry. To that extent at least, it should seem fairly clear why analytical progress might be valuable: it would include much illumination of the sort which is characteristic of ordinary scientific progress.

I will now argue that there are some crucial features of Wittgenstein's early view which nonetheless make the need for analysis rather hard to explain. First, Wittgenstein maintains that the sense of a proposition is completely determinate: that a proposition lays down a condition of agreement and disagreement which reality cannot evade. But second, thinking is subject to the standards of logic just inasmuch as it is the tokening of propositions. Together, these commitments combine for a seemingly wild implication: that there is no rationally responsive thinking but as a grasp of wholly determinate truth-conditions. The theory of propositions as pictures was supposed to account for this implication as a datum.

2.2.1 Determinacy of sense

Wittgenstein asserts the determinacy of sense in these remarks:

> Reality must be held by a proposition to yes or no.
> In order to do that, it must describe reality completely. [4.023a-b, my translation]

By "complete", Wittgenstein does not mean that the truth-value of one proposition determines the truth-value of all of them. Rather, in Tractarian parlance, a proposition is said to have a sense: it agrees and disagrees with possible states of the world. What's meant at 4.023 is a claim about sense: that while whether a proposition agrees or disagrees will depend on the facts, that it agrees or disagrees does not. Thus, the condition of agreement and disagreement is itself unconditional.

Wittgenstein distinguishes his view from those of Frege and Russell in its respect for the determinacy of sense. He maintained that Fregean thoughts are not relevantly determinate.[16] A plausible source of indeterminacy would be Frege's account of the contents of empty names or improper definite descriptions. For example, Frege is widely understood to have maintained that if a sentence contains an empty name in a nonoblique context, then it expresses a thought with no truth-value. Some commentators have volunteered this consideration as a subtext of 3.24 (see in particular Hart (1971) and Diamond **?**).

Wittgenstein's most explicit complaint about Frege's failure to respect the determinacy of sense is not this initially obvious one. At 4.063, he compares Frege's conception of thoughts as composite names to a specification of coordinates on a sheet of paper. Such a specification could be considered "true" if corresponding point on the paper is black, and false otherwise. But, he argued, just as specification of coordinates can be understood without knowing the colors black and white, likewise Frege's view allowed that a thought can be grasped without knowing what it would be for the thought to be true or false. So there is a cognitive gap between thought and truth-valuedness: somebody could be thought-wise and truth-blind. Correlative to the cognitive gap is a metaphysical one: on Wittgenstein's understanding, truth-valuedness is a contingent property of Fregean thought.[17]

In contrast, it would seem that nothing could respect the determinacy of sense better than the propositions described in early Russell (1903); see **?**, **?**. An early Russellian proposition would consist of some objects somehow related to each other. What would make it true is just that its objects are so related. So a proposition would just be what makes it true. A proposition would be no farther from its sense than it is from itself. Indeed, this would seem to be the only way for a proposition to determine its sense, granted the Russellian assumption that no relations are internal.

Wittgenstein found something to admire in Russell's approach, particularly in its partial assimilation of propositions to facts. As I'll spell out shortly, he worked to address Russell's worry that the view cannot account for falsehood:

> 3.13e In the proposition the form of its sense is contained, but not its content.

Nonetheless, Wittgenstein cannot be understood to allow that Russell's conception of propositions fully respects the determinacy of sense either. Wittgenstein maintains that fixing the sense of one proposition entails fixing its agreement and disagreement

[16]This is not to say that Fregean thoughts weren't supposed to be determinate. In the opening paragraphs of **?**, Frege advances a kind of open-question argument against a correspondence theory of truth, which appears to presuppose a version of sense-determinacy.

[17] As Proops (1997) observes, Wittgenstein seems to have had in mind that the mere coordinate specification of a place does not agree or disagree with the place specified; what could agree or disagree is the result of combining the specification with a predicate like "is black". Wittgenstein somehow seems to have found Frege's (1879) appeal to a truth-predicate to play an analogous role in the theory of judgment as stepping from a thought to its truth-value. See Proops (1997) and **?** for further discussion.

with distributions of truth-value over others. For example, it would be part of the sense of p to agree with those possibilities in which $p \wedge q$ is true, to disagree with the possibility in which $p \rightarrow q$ is true while q is false, and so on. A proposition's condition of agreement and disagreement with facts must be informed by its logical relationships with other propositions.

> 3.42a Although a proposition may only determine one place in logical space, the whole logical space must already be given by it.

Wittgenstein alleges that both Frege and Russell mistakenly suppose the truth-functional connectives to express something like ordinary relations, such as right and left (5.42). Accordingly, the application of connectives would always introduce new possibillities, whose logical interaction becomes mysterious (3.42). Consider a possibility in which p materially implies q. For Wittgenstein's Russell, about that possibility there remains a further question whether, say, $\neg p$ "disjoins" q. The two questions are settled by two facts. Nonetheless, the facts are clearly somehow related: the material implication of q by p does illuminates whether $\neg p$ disjoins q. But this inference from one to another requires appeal to the truth of a logical proposition. It does so happen that the logical proposition is true actually. Wittgenstein objects that on Russell's view, with logical truths bustling on all fours with the rest, it cannot be explained why the imagined world must have them in common with the actual one.[18]

2.2.2 Immediacy of sense

As we've just seen, Wittgenstein finds that a proposition has packed into it everything required for it to agree or disagree with reailtiy, no matter how reality has been fixed. but

I've now argued for two interpretive claims. First, the rational possibilities for thinking are to be explained by the structure of its propositional content. But second, it's in the essence of a proposition to fix a wholly determinate truth-condition. These claims combine to yield the promised wild result: thinking is rationally responsive just as traffic in what is wholly determinately truth-conditioned. Let's call this the immediacy of sense: that there is no cognitive play with propositions, which is to say no rationally responsive thinking, except in the grasp of truth-conditions which are wholly determinate.

Wittgenstein's commitments to the immediacy of propositions and to the determinacy of sense require a commitment to the immediacy of sense. Of course, interpretive fidelity is not closed under entailment. Maybe he never put two and two together? Evidence to the contrary appears in the the pre-*Tractatus* notebooks. For example,

> When I say this watch is shiny, and what I mean by this watch alters its composition in the smallest particular, then this means not merely that the sense of the sentence alters in its content, but also what I am saying about this watch straightway alters its sense. The whole form of the proposition alters. (Wittgenstein, 1979, 16.6.15). [more broadly see 15.6.15-22.6.15]

[18] Ricketts (1985, ?, ?) has urged that the problem here is a Carroll-style regress. But as this paragraph indicates, it's not clear that the situation allows a regress to get started.

So Wittgenstein did struggle with the immediacy of sense. But why? The commitments I've found to entail it are hardly incontrovertible. Why not just drop one of those? Wittgenstein did not just stumble into the immediacy of sense. It expresses the most famous doctrine of the *Tractatus*: that propositions are pictures of reality.

Elementary picturing. That the picture theory was designed to secure something like the immediacy of sense should of course seem obvious. As Sullivan (2001) observes, the comparison of propositions to pictures seems precisely best motivated as an attempt to characterize how a proposition shows people its sense. For it's heuristically plausible that, when a person looks at a successful picture, what's seen is not the picture but rather what is depicted. The motivation emerges clearly here:

> 4.0311 One name stands for one thing, and another for another thing, and they are connected together. And so the whole, like a living picture, presents the atomic fact.

So when at 4.5, Wittgenstein says that the general form of the proposition is

> *Es verhält sich so und so.*

the *so und so* can be glossed as a demonstrative. To grasp how things must be if a proposition is true, it suffices to see how their representatives figure in the proposition itself.

Of course, it is one thing to say that an analogy of propositions to pictures gives a nice heuristic. But Wittgenstein means more than that:

> 4.01 A proposition is a picture of reality.
> The proposition is a model of the reality as we think it is.

The identification of propositions with pictures is actually supposed to explain, through an an account of the function of pictures, how a proposition makes its sense known.

I'll now rehearse an account of the grounds of the immediacy of sense in the pictoriality of propositions, which is due in large measure to Ramsey (1923) and Sullivan (2001). In broad outline, the account is straightforward. To say that a proposition is a picture is to say that as names figure in a picture, so their bearers are said to figure in reality.

> 2.14 The picture consists in the fact that its elements are combined with one another in a definite way.
>
> 2.15 That the elements of the picture are combined with one another in a definite way, represents that the things are so combined with one another.

Now neither a picture nor a fact is just a bunch of items: rather, in each case the items are constituents of a structure. A structure actualizes a possibility. In particular, a picture actualizes a possible manner of figuring of its constituents. Another possible structure is the depicted fact. On pain of precluding falsehood, the two structures are not the same: the actualizing of a possibility for names, i.e., the mere tokening of a proposition, is not the same as the corresponding actualization of a possibility for things.

But the two structures have something in common, and this is what Wittgenstein calls pictorial form. For example, a blue block in front of a red one may depict a blue car in front of a red car. In this case, the color and spatial relationships of the cars are taken up into the picture itself, and therefore belong to its pictorial form. But material constitution does not pertain to this pictorial form, because the picture doesn't represent the cars as made of wood. The concept of pictorial form admits degrees of abstraction. The paint on each block can be replaced by labels which are colorless; the spatial relationship of the two blocks can be replaced by a typographically assymetric label with an end attached to each. Still, any pictorial result of such steps of abstraction retains something in common with the situation represented. That ultimate, inescapable commonality is what Wittgenstein calls "logical form".

At this point, Wittgenstein takes a decisive explanatory step:

> 2.033 Form is the possibility of structure.

Upon taking this step, it follows that at any level of representational abstraction, the form of a picture is the possibility both of the picture and of what it represents. Thus, the step amounts to identifying those possibilities as one. Now, if the possibility of one thing is the same as the possibility of another, then the actuality of one of them in some sense "contains" the possibility of the other. Therefore the actuality of a picture, say in the tokening of a proposition, contains the possibility of the fact that would correspond. It's by containing a possible fact in this way, that the picture shows its sense.

This story of the grounding of the immediacy of sense in the pictoriality of propositions can be summarized as follows. A picture contains the possibility it represents. Since propositions are pictures, a proposition is itself the grasp of a possibility.

What a proposition says. So far as it goes, I don't want to contest the Ramsey-Sullivan story. Rather, I'll argue that without further specification, the story doesn't secure the immediacy of sense of propositions which are nonelementary. The point is trivial: possibilities interact. Can the cat be on the mat while the mat hangs out to dry? A fact, as a combination of objects, has what let's call a modal profile: its obtaining either precludes or tolerates the obtaining of each other possible fact. The Ramsey-Sullivan story gives an appearance of explaining the immediacy of sense of elementary propositions because the modal profiles of atomic facts are trivial: each can be the case or not while all the others remain the same. What does it mean to say that profiles of nonatomic facts must be shared by their pictures? What can it mean for pictures to actualize possibilities whose profiles are nontrivial?

Suppose fact F precludes fact G. Presumably, each fact can be pictured. How does the capacity of picturing reflect the preclusion? The account of modal reflection suggested by Sullivan is that something can be representationally constructed only if it represents something metaphysically possible (see also **?**). Now way to extend this proposal to modal interaction would be to say that if F precludes G then a proposition that F obtains and a proposition that G cannot both be representationally constructed. But on at least one understanding, this does not work. For example, people can formulate contrary hypotheses, so that representations of incompossible situations are both constructed.

Another account of the formal commonality of facts and pictures might be this: the preclusion of G by F is reflected in that a picture of F can be true only if the picture of G is false. That a reflection of this sort must happen is of course highly plausible, but that's because it is vacuous. It is like saying that if two couples r and p meet and if r is married, then the marriage of r is mirrored by p in that the couple met by p is married.

I suggest that the more general commonality of facts and pictures can be derived from the heuristic that a proposition "constructs a world with the help of a logical scaffolding" (4.023). Namely, the profile of a fact would be taken up in what its picture depicts. For example, if fact F precludes fact G, then to say that F obtains is, among other things, to deny that G obtains. Conversely, if fact F tolerates fact G, then to say that F obtains is not thereby to deny that G obtains. So, the more generally required commonality is this: that for all F and G, the obtaining of F requires the obtaining of G iff to say that F obtains is thereby to say that G obtains. Let's call this the persistence of logical scaffolding.

To motivate the persistence of logical scaffolding, recall that the doctrine of determinacy of sense is that a proposition must by itself fix whatever a proposition's truth requires of the world. Now, suppose that P says that F obtains. And suppose that the obtaining of F requires the obtaining of G. Then the truth of P requires the obtaining of G. The determinacy of sense now implies that it must be given with P that if P is true then G obtains. It's this implication of the determinacy of sense that the persistence of logical scaffolding is meant to recognize. For what's a natural way in which it must be given with P that if P is true then G obtains? Simply that P says, among other things, that G obtains.

Thus, the persistence of logical scaffolding entails a particular conception of what a proposition says. By insisting that it's possible to draw inferences from a false proposition, Wittgenstein presupposes that logic contributes a standard of what a proposition's truth would require. Thus if Q logically follows from P, then the truth of P requires the truth of Q. In turn, all those requirements are packed into what a proposition says.

In the 5.1s, Wittgenstein begins to speak of what propositions affirm and deny (*bejahen* or *verneinen*). That language supports the present conception of what a proposition says. Let's say that P says exactly that F obtains, provided that P says that F obtains but also that whatever else says that F obtains also says whatever P says. Now an interpretation of Wittgenstein's language of affirmation and denial can be given like this. Suppose that Q says exactly that G obtains. Then P affirms Q if P says that G obtains. And P denies Q if P says that G doesn't obtain.

The concept of affirmation as characterized by the persistence of scaffolding yields a further elaboration of the remark that logical truths are tautologies. For if Q is a logical truth, then the truth of any proposition requires the truth of Q. So by the persistence of scaffolding, what affirms anything thereby affirms every logical truth. Or to put it another way, if anything has ever been said, then every logical truth has been said already.

I have now assembled the main ingredients of the problem Wittgenstein confronts in trying to characterize the kind of situation in which people have something to do with logic. People have something to do with logic just as there is need for the activity of analysis. Analysis amounts to redescribing possibilities so that their modal interdependence becomes manifest. If modal interdependence could become manifest, then it must possibly have been obscure. But how could modality be obscure? Any occurrence of a thought is the tokening of a proposition. And there can be no gap, indeed no epistemic gap, between a proposition and its truth-condition. I will now argue that for Wittgenstein, the obscurity of modality is an obscurity of a proposition in its tokens, the difficulty of seeing the symbol in the sign.

2.3 Sign and symbol

W compares language to clothing. Cf. Frege GL p vii

Wittgenstein realized that the very success of the theory of propositions as pictures should make that theory seem implausible:

> The proposition is a picture of reality.
>
> The proposition is a model of the reality as we think it is. [4.01]
>
> At the first glance the proposition—say as it stands printed on paper—does not seem to be a picture of the reality of which it treats. But nor does the musical score appear at first sight to be a picture of a musical piece; nor does our phonetic spelling (letters) seem to be a picture of our spoken language.
>
> And yet these symbolisms prove to be pictures—even in the ordinary sense of the word—of what they represent. [4.011]

Wittgenstein's account of the need for analysis opens by drawing a distinction between the appearance and the reality of propositions.

> The sign is the part of the symbol perceptible by the senses. [3.32]

In this remark, Wittgenstein proposes that propositions and propositional parts have a sensibly perceptible aspect: this is the concept of sign. A sign is therefore something like an appearance of a proposition. The concept of sign therefore assumes many of the ambiguities typical of the concept of appearance.

To simplify the discussion, let's assume that Wittgenstein understands the concept of sign to support a type-token distinction. Then, how are sign-types individuated? At one extreme, this may turn on what the token symbolizes. Let's say that a type is univocal if there's a symbol such that to be an occurrence of that type just is at least to belong to that symbol. Univocality is suggested by Wittgenstein's explanation of "propositional sign":

> The sign through which we express the thought I call the propositional sign. And the proposition is the propositional sign in its projective relation to the world. [3.12]

Thus, a propositional sign-type is a sign-type all of whose tokens belong to the same propositional symbol.[19] However, it is emphatically not the case that all sign-types are univocal:

> Two different symbols can therefore have the sign (the written sign or the sound sign) in common—they then signify in different ways. [3.321]

As is well known, univocality failures are crucial to Wittgenstein's explanation of the need for analysis. But how?

Let's say that a type is really equivocal if its instances actually don't all belong to the same symbol. What sign-types could sustain real equivocality? **?** proposes a notion of type whose individuation is rooted independently of participation in symbols. An example would be similarity classes of ink-patterns, such as the class of all phenomena relevantly similar to a Baskerville inscription of the first line of Gray's Elegy. Let's call Kremer's types "artifactual". It's a platitude, then, that artifactual types sustain real equivocality.

But recall that Wittgenstein introduces the sign-symbol distinction partly to explain the need for analysis. This explanatory strategy issues another constraint on the notion of sign-type. Consider, for example:

> In the language of everyday life it very often happens that the same word signifies in two different ways—and therefore belongs to two different symbols—or that two words, which signify in different ways, are apparently applied in the same way in the proposition. [3.323a] Thus there easily arise the most fundamental confusions (of which the whole of philosophy is full). [3.324]

Wittgenstein would surely want to allow, for example, that one "fundamental confusion" can be transmitted from a printed German book through an English lecture into the sympathetic mind of a monolingual student.

So, a sign-type that meets Wittgenstein's explanatory strategy must not only sustain real equivocation but also be reinstantiable through many formats. Or more strongly still, it looks as though the relevant notion of sign-type must be such that one item can belong to the same type as another provided merely that the two items can reasonably be taken, correctly or not, to belong to the same symbol. Let's say that such sign-types are communicatively robust. Then, it's doubtful tht thin notions of sign-type, like that of artifactual type, are communicatively robust. In that case, they couldn't do the required explatory work.[20]

I suggest that the relevant notion of sign-type may be generated by relationships of apparent cosymbolization. In a little more detail, say that an occurrence x of a symbol cosymbolizes with occurrence y if the symbol of x is the symbol of y. It's plausible to

[19] Conversely, it's not clear that Wittgenstein wants to maintain that no two propositional sign-types correspond to the same proposition. Perhaps this is suggested by the talk at 3.143 of a "printed proposition" (*gedruckten Satz*), but not decisively.

[20] A second difficulty with Kremer's proposal is that a huge variety of physical phenomena could in principle be used to symbolize. This includes phenomena which will never actually be encountered by any human being. Hence Kremer's account yields an unreasonable extension of the class of tokens of all sign-types.

suppose that often, an occurrence x is such that for some occurrence y, the symbol of x can be identified only by recognizing cosymbolization of x with y. Anaphora form a natural class of examples. In a broader range of cases, including quotation, transcription and translation, it's plausible that cosymbolizing relations are partly constitutive of which symbols occur.[21] These examples suggest that identifying the symbol of an occurrence often depends on general capacities to recognize occurrences as cosymbolizing. Now, consider two phenomena x and y which may or may not even be occurrences of symbols, let alone of the same symbol. Say that x apparently cosymbolizes with y if x has a propensity to be taken to cosymbolize with y.[22] Finally, an apparent symbol-type is a class of phenomena which apparently cosymbolize.

The notion of apparent symbol-type gives a good interpretation of Wittgenstein's concept of sign. First, it yields a reasonable extension for the class of all sign-tokens: namely, as just what people tend to take to symbolize. For by stipulation, something instantiates an apparent symbol-type iff people tend to take it to cosymbolize with something. But, everybody's tendencies reflect an appreciation that something symbolizes iff it cosymbolizes with itself. So, something instantiates an apparent symbol-type iff people tend to take it to symbolize.

Second, the notion of apparent symbol-type fulfills both of the constraints which Wittgenstein's account of confusion imposes on the concept of sign. On the one hand, it is clear that there are many pairs of items which people tend to take to cosymbolize but which don't cosymbolize in fact: for example, tokens of homophones. Any such pair will be an instance of real equivocation. But on the other hand, cosymbolization can be naturally recognized between tokens of different words, of the same word in different formats, and even of words in different languages. So apparent symbol-types that are communicatively robust in a way that underwrites Wittgenstein's account of confusion.

2.4 Who needs analysis?

An analysis-token institutes coordination of two tokens it contains. (Some affinity with Frege's begriffsschrift identity.) To the extent that the two tokens are themselves coordinated with other tokens, the analysis achieves a kind of (pre-semantic) generality.

Why are such stipulations needed? Equivocation.

Predications of color make a good case for the pervasive, multidimensional equivocation Wittgenstein envisions. Correctly or not,[23] Wittgenstein considers vagueness to be a kind of equivocation. And color predicates are vague: at the very least, there are borderline cases where people can't generally agree on whether, say, some token of "this is red" is true. But, the determinacy of sense implies that if that token of "this is red" does express exactly one proposition, then it draws a perfectly sharp distinction

[21] For example, a reader of the Ogden-Ramsey translation of the *Tractatus* might wonder whether two occurrences of "determination" belong to the same symbol; to this end it's a relevant consideration whether they translate the same German word.

[22] To be a little more precise: the notion of apparent cosymbolization has a suppressed parameter, representing features of the relata which trigger the capacities of cognizers to take the relata to cosymbolize. For example, a pronoun can cosymbolize with a proper name because of rules about anaphora, whereas the translation of a word cosymbolizes with the occurrence translated because of essential features of translation.

[23] **?**.

across all possible worlds. So, speaker and audience could (if theoretically inclined) agree that the token does draw such a sharp distinction, but they could not articulate that distinction in independent terms. Transposing a suggestion from the *Notebooks* (1979): somebody might point to the very subject of predication and say "the same color as this". But, understanding that occurrence of "same color" would presumably require the very discriminating capacity it's supposed to impart.

Color predicates equivocate not just between neighbors on a continuum, but also between lookalikes and concomitants from profoundly different categories. For example, Descartes contended that people habitually confuse events in a person and real qualities of a distal object; and perhaps for him even the very idea of a human being is a confused amalgam of thinking and extended natures. Contemporary philosophers of color would doubtless want to draw distinctions differently, but most would agree that the usage is equivocal.[24]

To say that expressions are equivocal is not call for disambiguation. In practice, enacting a segmentation of a physical continuum generally requires protecting intervening regions as "borderline".[25] Descartes contended that propensities to confusion have value: in the ordinary couse of things, it's beneficial for people to see berries as red, or to take some physical changes personally. At 3.24, Wittgenstein suggests that logical generality depends on an indefinite semantic state. These are all examples where an indeterminate relationship between sign and symbol doesn't in itself call for redress.

Shortly after introducing the sign-symbol distinction at 3.32, Wittgenstein describes propositions as determining positions in a "logical space":

> The proposition determines a place in logical space: the existence of this logical place is guaranteed by the existence of the constituent parts alone, by the existence of the significant proposition. [3.4]

The analogy between propositions and places may help to illuminate Wittgenstein's conception of the need for analysis. Of course, there's a question just how to spell out the notion of "logical place", which I'll address in the next section. But for now, note that if the analogy is applied to the need for analysis as sketched in the 3.32s, then it would compare needing to identify a proposition with needing to identify a spatial location.

What is the need to identify a logical or spatial place? Neither analysis nor geography is an exercise in the pure theory of internal or spatial relations. On the one hand, the 3.32s suggest that the need for analysis involves not simply propositions, but the appearance of propositions in signs. Analytical progress involves coming to understand one propositional sign by reference to another sign of the very same

[24]Of course, the notion of equivocality is equivocal itself. At a minimum, the equivocality of a sign would seem to imply that people wouldn't agree about how its uses are logically related to other uses of signs. However, this may be either because people customarily use the signs in ignorance of their precise logic, or because the signs somehow typically fail to determine a precise logic at all. Furthermore, it's not clear whether Wittgenstein maintains only that equivocality operates only on the level of types, or indiscernibility-classes of propositional simulacra, or whether even a single sign-token may belong to a range of propositions simultaneously. Regrettably, my purposes don't require settling these issues.

[25]?.

proposition. Only the signs change, but the proposition stays the same. Analysis then addresses problems like: "how is the proposition expressed by this sign related to the proposition expressed by those ones?" On the other hand, geography also addresses not mere geometry of physical space, but spatial relationships between its physical occupants. Correspondingly, a question of geography may take the form "how is the place occupied by this feature related to the place occupied by those ones?"

The comparison of logical and physical space begins to illuminate what distinguishes helpful and unhelpful analyses. Suppose that somebody doesn't know where her violin is. Does it help to learn that it's in its case? Perhaps, if she knows that the case is in the car. But if she has a very important violin, then the violin's location might be better known than the car's. More generally, it looks as though any correct statement of the location of her violin can be found unhelpful in some sufficiently pathological circumstance. So what does it take to help? Needing to know where something is, the violinist suffers from a kind of epistemic fragmentation. What relieves it could be anything that allows her to reintegrate what she knows about where her violin is into her systematic knowledge of the places of other features of the world. It's an analogous kind of cognitive fragmentation that a Wittgensteinian analysis is supposed to redress. Somebody can very well grasp that, e.g., "Tony is two sheets to the wind" is contradicted by "No, he isn't", even that for all n it's equivalent to ⌜Tony is n sheets to the wind⌝, and so on, yet still need an explanation of what any of them actually says. A whole system of signs forms a kind of cognitive raft, unmoored to the mainland.

Like needing to learn where something is, needs for analysis are highly heterogenous. One important range of cases is the following: people may have a deep understanding of the internal structures of each of two branches of inquiry, but also see that the two structures must be related in ways they can't grasp: quantum mechanics and general relativity, neuroscience and psychology, set theory and number theory. These are questions of what might very broadly be called conceptual geography; Wittgenstein's appreciation of their importance owes a great debt to Russell. But Wittgenstein was probably more troubled by the modernist predicament that people could, as it were, know all the geographical facts, but fail to know where they themselves are. Such an acknowledgement in the *Tractatus* of a subjective element in philosophical problems might be understood to anticipate a much later remark: "a philosophical problem has the form: I don't know my way around".

Although failures of understanding are as heterogeneous as failures to know where something is, in each case all of the various problems are supposed to have an essentially common feature. As I observed above, learning where something is amounts to integrating bodies of spatial representation. Such a demand for integration seems reasonable with respect to any spatially located phenomena. In writing the *Tractatus* Wittgenstein maintained that the logical places of propositions form a single system too.

2.5 The contours of logical space

In the previous section, I quoted the remark 3.4 which describes propositions as determining positions in logical space. But what sort of model of space did he have in mind? Sullivan (2001) suggests that there may be two, which correspond to the two notions of possibility from § ??.

The first spatial model dominates the opening of the *Tractatus* (see in particular 2.013 and 2.11). There, each dimension of the space comprises a range of objects of the same logical form. Possible states of affairs are points which result by choosing an object from each dimension. A selection of some points as "occupied" then determines a possible total state of the world.

A second model of logical space—let's call it the possible worlds model—gains ascendance at 4.63.[26] Here, each elementary proposition determines a distinct dimension whose values are the two truth-values. A point of the space is then a choice of truth-value for each elementary proposition. To propositions in general there correspond regions of the space, or sets of possible worlds.

The possible-worlds model gives a class-theoretic representation of logical relationships. The negation of some proposition p determines the complement of the region determined by p; and similarly a disjunction of several propositions determines the union of their regions. The region determined by a tautology is the collection of all points in the space, whereas a contradiction determines the empty region. What corresponds, in the model, to the entailment of q by p is that the region determined by q includes the region determined by p.

I want to focus on the role played by the possible-worlds of logical space in Wittgenstein's account of logic. As with the first model, its interpretation has occasioned much controversy. The general form of the controversy is "which is prior, the coordinates or the regions?" So-called "realist" or "atomist" interpreters, who take the coordinates to be prior, also tend to think that the opening of the *Tractatus* reports an ontology which determines what can be said in any possible language. States of affairs would form a self-standing grid, from which all possible senses must have been constructed, and in terms of which perspicuous expression can be regained. The holist perspective differs starkly. From this point of view, the opening passages present an edifying myth intended to convey how language works. Linguistic usage determines a system of logical relationships. Wittgenstein proposes clarifying the logic of usage by coordinatizing it against a set of independent elementary propositions, like a cartesian coordinatization of an antecedently determinate geometry.[27]

The two perspectives on the possible-worlds model share a background understanding. Upon analysis, propositions get rewritten in a new notation. Both readings assume that it's the possible worlds model which explains how the expressions of the new notation are logically related. I believe that this shared understanding does not cohere with the Wittgenstein's conception of analysis, as grounded in relationships between propositions which are internal.

To see the difficulty, suppose some propositions have been chosen as elementary ones, and that some methods have been given for constructing further propositional signs. Further suppose that each elementary proposition determines the region consist-

[26] As **?** observes, the earlier-quoted remark 3.4 can be understood in terms of either model, depending on whether "place" is interpreted as "point" or as "region".

[27] Reverting to §**??**'s further analogy with physical space: the atomists may be compared to Newtonians, in holding that objects or states of affairs absolutely identify some propositions as elementary. The holists might be compared with anti-Newtonians, maintaining instead that the elementariness characterizes only a structural role in the system.

ing of all and only those points at which it is true, and that each method of constructing further propositional signs is associated with a corresponding function on regions: negation goes to complementation, disjunction to union, and so on. None of this implies anything about which points, or even any points, exist.

Here it seems natural to consider the opening remarks of the *Tractatus*. The very first remark might be taken to say that at least one point exists, corresponding to what is actually the case. Remark 1.21 seems to amounts to saying that the space of possible worlds is closed under the procedure swapping the truth-value of an atomic proposition. If the number of elementary propositions is finite, this implies that the space of possible worlds does represent every truth-value assignment. But suppose that the number of elementary propositions is infinite. Suppose that an infinite disjunction of elementary propositions has infinitely many true disjuncts. Then for all that's been said so far, the infinite disjunction is true at every possible world. However, it's also consistent with the opening of the *Tractatus* that logical space is populated much more richly; then any disjunction of elementary propositions would have the possibility of falsehood.

At this point, there are two issues to be raised. First, suppose that logic really is to be explained by the possible worlds model. As we've just seen, that model leaves considerable leeway as to how logical space gets populated. This leeway generates divergent verdicts about the extension of logical relationships.[28] So under the hypothesis that the possible worlds model explains logic, it follows that those relationships depend on how logical space gets populated. But then it's not clear that the relationships can after all be understood as internal.[29]

Second, it seems natural to conclude that logical space, if populated as sparsely as the opening of the *Tractatus* could possibly allow, would give incorrect verdicts about the logic of infinitary disjunctions. But what makes those verdicts incorrect? It looks as though there is some other source of requirements which an instantiation of the possible worlds model could be said to respect or violate. In the rest of this paper, I'll argue that Wittgenstein envisioned that in becoming propositions, signs do issue such independently constituted requirements.

This conclusion will cut against the atomist and holist readings somewhat differently, because they have slightly different conceptions of the determinacy of sense. For the atomist, elementary propositions are like unit vectors in Newtonian space, an objective and universal set of coordinates. This yields an account of the determinacy of sense which is independent of the full consequence relation: that each proposition specifies a truth-value relative to given possible world. But by taking truth to be prior to consequence, the atomist must understand logic to inherit the arbitrariness endemic to the possible worlds model.

The holist retorts to the atomist that the possible worlds model is only a guide to a finding a perspicuous notation. Logical connections don't need to be constituted by the possible worlds model, because they are already immanent in customary usage. On the account of **?**, elementary propositions inhere in a "terminus ad quem" of analysis,

[28]Now, it might be protested: "obviously the spare interpretation is not what Wittgenstein intended! Clearly he had in mind the full power set." I'll address this issue in § 4.2.2.

[29]This point resembles an observation of **?**.

a moment of having perspicuously rewritten all propositions whatsoever. But holist still owes an account of how the new notation makes anything perspicuous. Like Ricketts, at this point they may fall back onto the possible worlds model.[30] But then the notation can be clarifying only if the model determines a consequence relation univocally. And it doesn't. Hence, the tenability of the holist commitment to the immanence of logic within language devolves on an independent and unambiguous source of logical requirements.

3 A description of a sign-language

So, I've argued that Wittgenstein advocates a reorientation of logic. He finds Frege and Russell to hold that logic aims at discovering a body of truths. But instead, logic is, or ought to be, an activity of clarifying relationships between possibilities. Still, Wittgenstein certainly allowed that the logicists made great advances despite allegedly organizing of logic around a misinterpretation of a degenerate case. Frege's account of the general form of the theorem represents proofs as concatenations of steps, each step being underwritten by antecedently accepted laws. In turn, the applicability of a law to some nodes of a proof requires those nodes to be represented at least as though they had some intrinsic structure. For Wittgenstein, the notation assumes a broader explanatory task: for example it needs witness relations not just of the form "q is logically true if p is" but also "logically, q is true if p is", and so on. The signs need to witness all truth-functional interdependence.

In the *Tractatus*, Wittgenstein describes such a notation like this:

> Now it seems possible to give the most general propositional form; i.e., to give a description of the propositions of *any one* sign-language [irgend einer *Zeichensprache*], such that every possible sense can be expressed by a symbol which satisfies the description, and every symbol which satisfies the description can express a sense, provided that meanings of names are suitably chosen. [4.5a, my translation, emphasis in original]

This is a remarkable fountain of scope ambiguities, centering on the partly emphasized phrase "irgend einer *Zeichensprache*". The phrase *irgend einer* might be read existentially, so that the GPF would describe the propositions of some one arbitrarily chosen sign-language. Or it might be rendered universally;[31] this suggests rather that the GPF simultaneously describes the propositions of all sign-languages. Circumstantial and conceptual considerations favor the existential reading.

First, Ogden-Ramsey translate *irgend einer* as "some one sign-language". This rules out the narrow-scope universal reading, because something that describes some one language needn't (at least not trivially) describes every language. Similarly it rules out the wide-scope universal reading "for all languages L and for all descriptions D, D is the GPF iff…".[32] However, Wittgenstein exchanged detailed correspondence with

[30] An even more extreme holist deference to the model appears in Zalabardo (2010).

[31] Pears-McGuiness 1961 use "*any* sign-language *whatsoever*", which retains the verbal ambiguity of the German.

[32] Because $\forall y \forall x(Gx \leftrightarrow Rxy)$ implies $Ga \rightarrow \forall y Ray$.

Ogden about the translation. He makes three comments about the rendering of 4.5, but leaves alone the rendering of *irgend einer*. Since that rendering is unambiguously existential and 4.5 is a an obviously crucial remark, it's likely he'd have said something.

Also more substantial considerations supporting the Ogden-Ramsey rendering. Suppose to the contrary that *irgendeiner* is rendered as "every", so that the GPF gives a description of every sign-language whatsoever. Now, a purely syntactical description of all sign-languages including Esperanto and Semaphore would seem to be out of the question. Hence the universal reading presumably implies a weaker notion of "description", according to which the GPF's "describes" a language only by specifying the totality of possible truth-conditions (of sentences of any language at all). However, then it is odd that Wittgenstein explicitly says that what fits the description are not senses but symbols. Moreover, the final clause about meaning-assignments strongly indicates that (i) what satisfies the description contains names which can be assigned meanings, but (ii) whether or not something satisfies the description doesn't depend on what names have been assigned the meanings. That is, the final clause ought to fortify Wittgenstein's snipe at Russell's type theory: "It can be seen that Russell must be wrong, because he had to mention the meaning of signs when establishing the rules for them" (3.331a). So there are reasons to reject the weaker reading of "describes" which is forced by universal reading of *irgendeiner*.

In sum, the first sentence of 4.5 should be parsed like this:

> something, D, is the most general propositional form iff
> there is a sign-language L such that
> what satisfies D are precisely the symbols of L, and
> when meanings are suitably assigned to the names of L, then
> each symbol of L expresses some sense, and
> every sense is expressed by some symbol of L.

So the most general propositional form is supposed to describe some one sign-language. This described language would be "universal" in that given an assignment of meanings to its constituent names, what is expressed by its propositional signs is the absolute totality of what can be said or thought.[33]

Of course, many sign-systems are equivalently universal. Nothing essentially privileges the one Wittgenstein happens to choose. Wittgenstein worries that the necessarily arbitrary choice of one system raises a danger that accidents of this choice get mistaken for what is logically essential (3.34ff). For example, Wittgenstein seems to think that Frege and Russell did this when they "introduced generality in connection with logical product or logical sum" (5.521). Failure to detect an arbitrary choice here leads to what by Wittgenstein's lights is the mistaken idea that quantifiers are (higher-level) predicates. So he finds it worth emphasizing that the language described is *any one* among many which would in principle, if not in practice, serve just as well.

Now, Witgenstein undertakes in the *Tractatus* 5s to sketch a description of the sort 4.5 promises. The description coalesces in the 5.5s and culminates at T6. At 6.124 he alludes back to 4.5 and mentions an upshot of the development:

[33] Thanks here are due to — and — for discussion, though I don't claim these authorities to support the present reading.

> [...] if we know the syntax of any one sign-language [*irgendeiner Zeichensprache*], then all the propositions of logic are already given.

As I've urged, this would be a mere corollary of the main result, an account of propositional structure in which all truth-functionality would be rooted. In the rest of this section, I will sketch Wittgenstein's attempt to fulfil the promise of 4.5.

The construction of a sign-system falls into two stages:

> Suppose *all* elementary propositions were given me: then we can simply ask what propositions I can build out of them. And these are *all* propositions and *so* are they limited.

This remark distinguishes what's "given" and what's properly "constructed". I'll argue that what's "given" is primarily a system of signs of the totality of elementary propositions, and what's constructed is primarily the signs of all other propositions. Each constructed sign should arise from what's given, by iterated syntactical combinations which are purely finitary. This method of construction will ensure that the truth-values of constructed propositional signs depend only on the truth-values of elementary propositions.

3.1 Elementary propositions

Wittgenstein clearly holds that the general propositional form can be given *a priori*, and that this amounts to describing how propositions are constructed from elementary ones. This would naturally suggest that the general form of an elementary proposition can be described in advance too.[34] But after voicing this expectation at 5.55, Wittgenstein replies that since elementary propositions consist of names and the number of names with different meanings cannot be given, therefore the composition of elementary propositions cannot be determined. He seems to conclude that since we cannot give the elementary propositions *a priori*, the attempt to do so must lead to obvious nonsense (5.5571).

As is well known, Wittgenstein does not treat elementary propositions as unstructured points. With elementary propositions in mind, he writes that "the proposition is articulated" (3.141); it is a nexus or concatenation of names (4.22). One piece of evidence that elementary propositions consist in part of names is the existence of propositions which are general: "the comprehension of the general propositions depends *palpably* on that of the elementary propositions" (4.411). In particular, the truth-value of a general proposition depends on the truth-values of many propositions, those whose sense have a "common mark" (3.317). This common mark is an "expression", the result of turning some constituents of a proposition into variables. I understand Wittgenstein to hold that a proposition is marked by such an expression iff the expression can be obtained by turning into variables some of that proposition's constituents.

So people are supposed to have some concept of elementary propositions, as constituted at least in part by names. But this inkling doesn't decide whether some elementary propositions contain twenty-seven names (5.5541). More generally, people's

[34] Indeed, Wittgenstein would seem to have expected as much throughout 1916: [[quote!]] 16.4.16 and 23.11.16.

understanding of language doesn't make obvious to them how names are combined into elementary propositions. For example, if some names constitute a proposition, can they do so in more than one way? How does the number of ways of combining names into a proposition compare with the number of ways of arranging them in a linear order? Such questions could only be answered at the end of analysis.

Wittgenstein's sharpest expression of the character of elementary propositions might be this:

> The essential nature of the propositional sign becomes very clear when we imagine it made up of spatial objects (such as tables, chairs, books) instead of written signs.
> The mutual spatial arrangement of these things then expresses the sense of the proposition. (3.1431)

This heuristic underlies Wittgenstein's conception of how names characterize the sense of elementary propositions. A proposition depicts a state of affairs. As names figure in the proposition, so their bearers are said to figure in the depicted state. Given any elementary propositions P, Q and names a, b, it can be asked: is how a figures in P just how b figures in Q? Likewise this can be asked of P, Q and two sequences of names. The characterization of the sense of propositions by names can be understood to consist in the answers to all such questions.

Such a characterization of the sense of propositions by nominal constituents can be understood to underly Wittgenstein's account of generalization over objects. To anticipate somewhat: a name together with a propositional sign determines a manner in which a name significantly figures, and thereby yields an indication of the class of those propositions in which some name or other figures likewise.

3.2 Nonelementary propositions

Let's now turn to the second half of Wittgenstein's description of a perspicuous, universally expressive sign-system. Given the elementary propositions and the characterization of their senses by names, it remains to ask: "what further propositions can I construct out of them?" To avoid getting bogged down in details I will present it in three successively finer approximations.

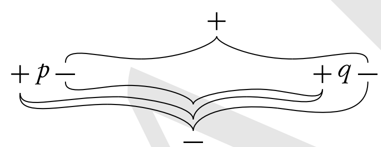
An account of this construction needs to satisfy three central constraints. First, the propositional notations have got to be designed so as to underwrite the thesis of *truth-functionality*: it must merely in virtue of the nature of formal relations between signs that the truth-value of a proposition depends only on the truth-values of elementary propositions. Second, Wittgenstein takes seriously the logic of quantifiers and even of mathematical induction. So we also have a constraint of *potentially transfinite support*: the account cannot rule out that a proposition's truth-value depend on the truth-values of infinitely many propositions. But third, the reference to what "I" can construct, or to how "we" people picture facts to ourselves, has got to be taken seriously. Let's call this a constraint of *finite realizability*: propositional signs must be capable of being written down by something like a human being, and the truth-functionality must appear in the resulting signs in a way that's available to human discernment. In the rest of this subsection, I'll sketch Wittgenstein's attempt to meet these three constraints, focusing especially on the second and third and deferring discussion of the first to §[[ref]].

3.2.1 Nonelementary propositions, a first pass

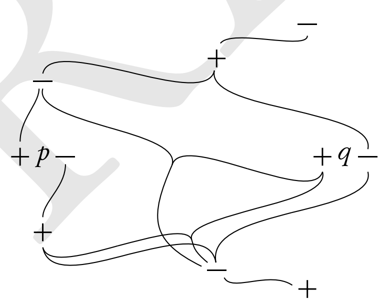
[[Should this go back to §2.2.1?]]

GPF versus axiomatization of logic (last chunk of diagram)

At 6.1203, Wittgenstein remarks that if a proposition contains no generality, then an intuitive method (*anschaulichen Methode*) suffices to determine whether or not it is a tautology. The method is to rewrite a proposition in a symbolism which Wittgenstein found already in (1913); like sense-tables, this seagull-notation should make truth-functionality recognizable. The method begins with the thought that a proposition has two "poles", that of truth and falsehood. I'll denote these by + and —. Now the result of a truth-operation gets rewritten as a rewiring of the poles of the bases of the operation. [[McGuinness]] In particular, joint negation wires + to —⋯— and — to everything else:



Wittgenstein conceives such a flight of seagulls to present a proposition as a result of a truth-operation on p and q. The presentation should therefore be iterable. For example, a material conditional would be written



Wittgenstein proposes to determine which propositions can be constructed by explaining how they're constructed. And his understanding of the means of construction looks pretty simple. Imagine you are sitting with some people around a table, engrossed in conversation. You can say whatever you want, only your speech-act needs to take one of two forms: you can assert an elementary proposition, or you can deny a bunch of stuff that has been said already. Denying some stuff already said would just be applying Wittgenstein's basic truth-operation. At a first pass, the idea of the general propositional form is that even under this constraint eventually everything could be said, provided that enough people spoke for long enough. As will become clear, this could only be a first pass. In the rest of this subsubsection, I will sketch its textual background.

The tabular notation of the *Tractatus* simply regiments those seagulls. For example, the diagram [[ref]] can also be written

p	q	
+	+	−
+	−	−
−	+	−
−	−	+

Such a table, like the seagull-assembly, expresses a proposition. Its top row contains the expression of further propositions; these are the *truth-arguments* of the table so presented (5.101). Each row of T's and F's below the series of truth-arguments signifies [*bedeuten*] a truth-possibility for those truth-arguments (4.31). It should then be self-explanatory what is said by the result of appending either a T or an F to each presentation of a truth-possibility.

Such a "sense-table" presents a proposition as a result of a single truth-operation. But like the seagull-method, the tabular construction can be iterated as well. For example, a material conditional could be expressed by iterating sense-tables like this:

<table>
<tr><td><table>
<tr><td>q</td><td><table><tr><td>p</td><td></td></tr><tr><td>+</td><td>−</td></tr><tr><td>−</td><td>+</td></tr></table></td><td></td></tr>
<tr><td>+</td><td>+</td><td>−</td></tr>
<tr><td>+</td><td>−</td><td>−</td></tr>
<tr><td>−</td><td>+</td><td>−</td></tr>
<tr><td>−</td><td>−</td><td>+</td></tr>
</table></td><td></td></tr>
<tr><td>+</td><td>−</td></tr>
<tr><td>−</td><td>+</td></tr>
</table>

Following conventions summarized above, this nesting of tables would be abbreviated as $(FT)((FFFT)(q,(FT)(p)))$ and then as $\mathrm{N}(\mathrm{N}(q,\mathrm{N}(p)))$.

It should be clear that the tables in the *Tractatus* are not truth-tables. Classically, a truth-table is an axiom of a semantic theory. A truth-table axiom could be reformulated in words like this: "if A and B are either true or false, then $N(A,B)$ is true if both A and B are false and is false otherwise, for all A and B." So a truth-table mentions, or quantifies over, formulas whose semantic values it describes. Now, it is true that Wittgenstein's tables are also to impart understanding of symbols. But Wittgenstein's tables do not mention, let alone generalize over, the symbols they treat. Rather, they just re-instantiate those same symbols, in a more "perspicuous" style.[35] The tables in the *Tractatus* aren't explanations of other symbols; they are symbols so reconstituted as to become self-explanatory.

[35] See, for example, Sullivan "the totality of facts". [[cite]]

Finally, Wittgenstein applies one more series of contractions. If the truth-possibilities for k truth-arguments are assigned some conventional order, then the the tabular expression of a truth-function can compressed into a list of 2^k T's and F's followed by a list of those truth-arguments. In a little more detail, Wittgenstein explains the universal truth-operation in the following way:

> Every truth-function is a result of the successive application of the operation '(— — — — —T)(ξ, ...)' to elementary propositions.
>
> This operation denies all the propositions in the right-hand pair of brackets and I call it the negation of these propositions. [5.5]

So the material conditional becomes [[]] In turn, the notation $\mathrm{N}(\cdots)$, or (— — — — —T)($\cdots$), then abbreviates a table in which only the last row ends with a T. And we end up with the expression $\mathrm{N}(\mathrm{N}(q, \mathrm{N}(p)))$.

So understood, a nonelementary propositional sign has a pointer-verdict structure. The pointer $(\cdots)$ presents signs of some other propositions and the verdict N is that those (other!) propositions are false. Here's a first pass at the syntax:

- Every sign of an elementary proposition is a propositional sign
- If $A_1, \ldots, A_n$ are propositional signs, then $(A_1, \ldots, A_n)$ is a bracket-expression
- If B is a bracket-expression, then NB is a propositional sign

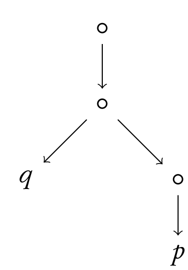
We can now say that a bracket-expression $(A_1, \ldots, A_n)$ presents B iff B is one of the A_i. Finally, let's say that $\mathrm{N}(\cdots)$ *directly denies* the signs presented by $(\cdots)$, and let's write $B \prec A$ to mean that A directly denies B.

Then for example,

$$\mathrm{N}(q, \mathrm{N}(p)) \prec \mathrm{N}(\mathrm{N}(q, \mathrm{N}(p)))$$
$$q \prec \mathrm{N}(q, \mathrm{N}(p)) \text{ and } \mathrm{N}(p) \prec \mathrm{N}(q, \mathrm{N}(p))$$
$$p \prec \mathrm{N}(p).$$

It's clear that the relation $\prec$ is wellfounded on the class of formulas introduced so far. And corresponding to each formula it generates an associated tree of denial. Continuing with the example above,

$\mathrm{N}(\mathrm{N}(q, \mathrm{N}(p)))$ $\rightsquigarrow$



It should be clear that the truth-conditions of formulas of ordinary truth-functional logic can all be represented by such trees of denial. Conversely, a finite tree of denial can express only the truth-condition of a formula of finitary truth-functional logic.

3.2.2 Nonelementary propositions, second pass

Obviously, an account on which propositions are constructed through trees of denial will respect the constraint of truth-functionality. However, Wittgenstein doesn't want to rule out that a proposition might be a truth-function of infinitely many elementary propositions. And even if somebody insisted that the number of objects is finite, the number would still seem to be rather large. This is no problem for the account sketched above. However, there was a further constraint, that propositions have to be capable of being written down by something like people. The naïve picture of just writing down the tree obviously can't satisfy all three constraints at once.

Wittgenstein did tried to reconcile truth-functionality and potentially transfinite support with the further constraint of finite realizability. Recall that we've supposed the operator-sign N to attach to a string of the form $(A_1, \ldots, A_k)$ where $A_1, \ldots, A_k$ are themselves propositional signs. But after introducing that operator, Wittgenstein elaborates:

> When a bracketed expression has propositions as its terms—and the order of the terms inside the brackets is indifferent—then I indicate it by a sign of the form '$(\overline{\xi})$'. 'ξ' is a variable whose values are terms of the bracketed expression and the bar over the variable indicates that it is the representative of all its values in the brackets. [[cite]]

From a logico-grammatical perspective, a bracket-expression contributes a suitable target of joint denial, namely a multiplicity or class-as-many of propositional signs. One way to do this is simply to include the multiplicity outright. But, there are alternatives. Wittgenstein proposes that an expression might act logically as though it were a multiplicity of propositional signs, yet syntactically be something else.

Within the brackets, Wittgenstein proposes to treat a propositional variable with a bar over it as though it were the multiplicity of propositional signs which the variable indicates. Thus, we are supposed to distinguish the syntactical contents of the bracket-expression (a barred propositional variable), from their logical contents, the multiplicity of signs indicated. A result of attaching an operator N to a bracket-expression directly denies those propositional signs which are the bracket-expression's logical contents. Syntactically, the proposal is to add to the previously enumerated formation rules a further one:

- If B is a propositional variable, then $(\overline{B})$ is a bracket-expression.

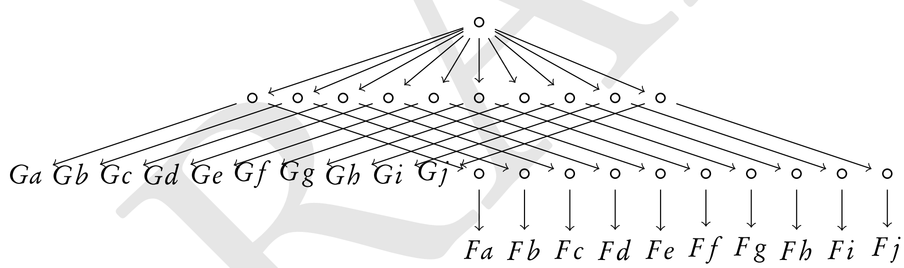
As before, $\mathrm{N}(\cdots)$ directly denies what $(\cdots)$ presents. But the analysis of presentation now needs to handle propositional variables:

- if $\cdots$ has the form $\overline{B}$, then $(\cdots)$ presents A iff B indicates A.

As before, the propositional signs will determine trees of denial. But now the trees may be larger than the signs; indeed as we'll see, it's even compatible with the three constraints on the sign-system that propositional signs determine trees of transfinite width and height.

But first we need to determine what ranges of propositional signs can be indicated by a propositional variable. Wittgenstein is lamentably vague about the syntactical construction of variables. So interpreters need to begin with what we know he intended variables to do, and reverse-engineer their syntax from that. Wittgenstein says that a variable is an expression which presents something propositions have in common, and thereby indicates those propositions [[cite]]. He clearly takes as a paradigm the internal mechanics of a quantifier. For Wittgenstein, the concept of quantifier really combines two pieces of machinery: the generality proper, which specifies some multiplicity of propositions as *all* the values of some propositional variable, and a verdict to the effect they're all true, not all false, or whatever. In turn, the propositional variable is something like a Russellian propositional function, and it would indicate the range of its values.

Consider, for example, a statement of the form "all Fs are Gs". This can be understood to deny, for each object o, that o be F but not G. In turn, to affirm that o is F but not G is just to deny jointly that o is G while o is not F. So, the analysis requires a variable whose values are signs of the form $\mathrm{N}(Go, \mathrm{N}(Fo))$ for any o. Wittgenstein doesn't give an explicit notation for this. But a widely favored approach is simply to write $x\mathrm{N}(Gx, \mathrm{N}(Fx))$. This can be understood as the result of prefixing x to the result of turning o into x in $\mathrm{N}(Go, \mathrm{N}(Fo))$.[36] Thus the original "all Fs are Gs" gets assimilated under the general propositional form by a notation which determines a tree of denial like this: [[insert formula into diagram]]



Of course instead of just $a, \ldots, j$, there are presumably more names than that. A node of a tree of denial could have many or even infinitely many children.

Although propositional functions are a paradigm propositional variable, Wittgenstein understood the resources of logic to outrun their capacities. Frege's 1879 famously culminates in an analysis of the concept of ancestral of a relation. Informally, the ancestral of a relation R is the relation of being connected by R. Given some relation R, Frege said that a property F is R-hereditary if Fs only bear R to Fs. Then he proposed b is connected to a by R if b has all of a's R-hereditary properties. By way of application, Frege **?** defines a relation of "successor" and then defines the natural numbers as those logical objects which are connected by the the successor relation to a logical object which is initially given.

Wittgenstein clearly thought that the ancestral was just as much a logical notion as are the quantifiers. But he had two complaints about Frege's treatment. First, he

[36] The xA notation slurs over a detail, since it leaves unspecified which constituent has been turned into a variable.

objected that Frege's analysis "contains a vicious circle" [[cite]]. Presumably he has in mind that Frege's analysis is sound only if the property to be defined, of being connected to a given object, is antecedently supposed exist (among the hereditary properties). The second objection addresses application. Wittgenstein denied that there are any logical objects. Rather, he envisaged applying the ancestral and related ideas to represent formal features of propositions about ordinary worldly stuff. So for example, "Barb is Lucy's ancestor" might be analyzed as

$\bigvee$(Barb is Lucy's parent,
  Barb is Lucy's parent's parent,
    Barb is Lucy's parent's parent's parent,
      …).

Assimilated under the general propositional form, this statement might determine a tree of denial whose top looks like [[tree of countable disjunction here]] where in turn T_n is the tree [[tree of a, b are R-connected by a chain of length n]].

To accommodate such analyses, Wittgenstein introduced a second kind of propositional variable, based on the mathematical idea of an infinite continued series.[37] For example, although the base e of the natural logarithm is transcendental, it can be expressed as the result of adding one to the sum

$$\sum\left(\frac{1}{1}, \frac{1}{1\times 2}, \frac{1}{1\times 2\times 3}, \ldots\right).$$

It is clear that something like a formal procedure generates each noninitial term of the sum from its predecessor. Wittgenstein supposes that "Barb is Lucy's ancestor" is a bit like e. Given only the primitive relation of parenthood, the relation of being an ancestor transcends first-order logic. But it can be analyzed as the disjunction of an infinite series, whose noninitial terms are the results of repeatedly applying a formal procedure.

More generally, Wittgenstein proposes that a propositional variable whose values are an infinite series of propositional signs can be given by specifying an initial term of the series plus a formal procedure for generating successor terms. He suggests the notation $[A, \xi, O(\xi)]$ where A is the initial term and $\xi \mapsto O(\xi)$ is the generating procedure. But he gives no notation for the procedures O, nor even much guidance about just which procedures might be considered "formal". Luckily the main purpose of this paper doesn't require much definiteness here.

We can now integrate these two kinds of variable into the sketch of the general propositional form. The expression of a propositional function involves "turning a propositional constituent into a variable". Suppose that A is a propositional sign and that a is a constituent of A. Let x_A be the first name-variable which doesn't occur in A. Now let's adopt a metalinguistic abbreviation: write aA for the result $x_A A[a/x_A]$ of prefixing x_A to the result of everywhere turning a into x_A in A. Then…

- if A is a propositional sign, then aA is a propositional variable.

[37] For this usage of the term *Formenreihe* see (Hilbert, 1912, 116).

The sign aA presents the manner in which the name a figures in A, and thereby indicates those propositions wherein something so figures. More explicitly, we may stipulate that aA indicates B iff there's a b such that $bB = aA$. Under natural conditions, it follows that aA indicates B iff there's a b not in $A[a/x_A]$ such that $A[a/x_A][x_A/b] = B$. It can thus be shown that the system which results from the rules of formation and evaluation so far developed is expressively equivalent to the "tractarian first-order logic" of Rogers and Wehmeier.[38]

Explicating the second kind of propositional variable requires explaining Wittgenstein's unexplained formal procedures and giving a notation for them. In this paper, I will keep out of that mess, leaving the formation rule schematic:

- if ξ, B is a sign of a formal procedure and A is a propositional sign, then $[A, \xi, B]$ is a propositional variable.

Let's define the nth term $[A, \xi, B]^n$ of the series of formulas indicated by $[A, \xi, B]$ in the following way:

- $[A, \xi, B]^0$ is A, and
- $[A, \xi, B]^{n+1}$ is the result of applying ξ, B to $[A, \xi, B]^n$.

We can now stipulate that $[A, \xi, B]$ indicates C iff C is $[A, \xi, B]^n$ for some n.

[[REDEFINE $\prec$]]

[[summary]]

3.2.3 Nonelementary propositions, third pass

We've now got a fairly detailed, if still slightly schematic reconstruction of the sign-system Wittgenstein announces at 4.5 and sketches at 5.501. However, there are still a couple of wrinkles. These involve Wittgenstein's proposal to write a bar over the propositional variables as they occur within square brackets. This was supposed to mean that the barred variable "stands for *all* its values". In this subsection I will indicate a technical difficulty with the reconstruction just sketched which in turn motivates an interpretation on which the bar plays an important disambiguating role.

Wittgenstein's bar notation fairly enough seems to mystify many commentators. The main problem is, what's the point? On one reading of its explanation, the point might be that the barred variable stands for "all" of its values rather than just for some of them. But Wittgenstein doesn't otherwise talk about restricting the range of a variable. Moreover, if the range of a variable could be restricted in one way, then it could presumably also be restricted in many different ways, so that the burden of disambiguation ought to fall on the restricted rather than the unrestricted form. So this reading doesn't make sense. [[Hochberg, 273ff]]

A more plausible reading is simply that the bar serves to distinguish between bracket-expressions which contain their terms syntactically from those bracket-expressions whose terms are merely indicated. But on the above reconstruction, still no ambiguity would arise if the bar were simply erased from all formulas.

[38] See appendix [[do it]] for a proof.

A yet further reading stems from the fact that when Wittgenstein discusses the analysis of quantification at 5.501 and the 5.52s, he doesn't clearly spell out how name-variables would receive indication of scope. For example, something needs to distinguish $\mathrm{N}(\overline{x\mathrm{N}(\overline{yRxy})})$ from $\mathrm{N}(\overline{y\mathrm{N}(\overline{xRxy})})$. But Wittgenstein doesn't mention the prefixes of x and y. Perhaps the bar is supposed to give an alternative? However, it clearly doesn't suffice to write for example $\mathrm{N}(\overline{\mathrm{N}(\overline{Rxy})})$. Rogers and Wehmeier (2012) suggest that Wittgenstein might have intended to delineate the scope of name-variables by vertically interleaving bars with circumflexed name-variables. For example, $\mathrm{N}(\overline{R\hat{y}})$. Although this ingenious suggestion does supply a purpose for the bar, it's not clear how it's supposed to improve upon the prefix-and-parenthesize approach of Peano and Russell. Furthermore, Wittgenstein seems fairly clear that the bar should be written over all propositional variables including also form-series variables, so the Rogers-Wehmeier explanation seems to operate at the wrong level of generality.

At this point it's tempting to conclude that the bar is just a grammatical nicety without any disambiguating power. But that would be disappointing because Wittgenstein otherwise goes to such lengths to strip the system's moving parts down to the bare minimum. Happily it turns out that the system as sketched above doesn't quite work under any definite realization I can see, and that it can be fixed only by introducing something precisely like the bar notation.

To see how the issue arises, note that Wittgenstein would surely want to allow full interaction of form-series variables and quantifiers. For example, we might want to say not just that Carol is a common ancestor of Bob and Alice but also merely that Bob and Alice have a common ancestor. It would be nice if that quantifier reached directly into a couple of countably infinite disjunctions (there is an x such that either x is a parent of Alice or a grandparent of Alice or..., and either x is a parent of Bob or ...). However, actually in the system as described above, instead what occurs in the scope of the $\exists x$ is a form-series variable. And there is no reason to think that the sign of the form-series variable contains an occurrence of x which corresponds exactly to its free occurrences in the form-series expansion.

Now, the problem is to ensure that quantifying into the context of a form-series variable works like ordinarily quantifying into the joint denial of the corresponding series of formulas. Now the natural response is to try fine-tuning the order in which a propositional sign gets expanded into a tree of denial. In the case under discussion, the solution is to expand form-series variables first and then expand quantifiers. To exert such fine-tuned control, a bar notation is precisely the ticket. Being written above the formula rather than within it, the bar notation can escape the usual order of the tree! Thus the order of the expansion of the variables can be determined, independently of formation order, by the vertical arrangement of the bars.

For example, an expansion which, as before, follows the construction tree of the propositional variables would be indicated by

$$\mathrm{N}(\overline{x\mathrm{N}(\overline{[A,\xi,B]})}).$$

To generate the tree of denial you work your way down through the stack of bars.

Then the first step gives

$$\mathrm{N}(\mathrm{N}(\overline{[A,\xi,B]})[x/a],\mathrm{N}(\overline{[A,\xi,B]})[x/b],\ldots).$$

Here as before, the outer quantificational x gets expanded first, affecting in an implementation-sensitive way the sign $[A,\xi,B]$. This is the approach which doesn't work in general, because it's not guaranteed that the substitution of a for x in a sign of a formal procedure coincides with substitution of a for x in the results of the formal procedure's application.

But, you might also write

$$\mathrm{N}(\overline{x\mathrm{N}(\overline{[A,\xi,B]})}).$$

Working down through the stack, now the form-series variable gets expanded first, this time returning

$$\mathrm{N}(\overline{x\mathrm{N}([A,\xi,B]^0,[A,\xi,B]^1,\ldots)}).$$

Here, the outer quantificational x has a pure countable disjunction of first-order formulas to act on.

More generally, the idea is to begin with the system of formulas generated by [[the rules of the previous section]] **??**, but then allow vertical rearranging of the bars. To generate the tree of denial, evaluate the bound variables following the partial order indicated by the bars' vertical arrangement. In practice, it may suffice to consider only formulas wherein all the form-series variables are barred above all the quantificational variables.

This refinement does introduce one complication. We are still working only with formulas that can be written down in finite space. But, the tree of denial will itself contain infinitary formulas. So we're not guaranteed that the evaluation procedure corresponds to a direct denial relation on the class of finitary formulas. In practice, some implementations may have the property that each node of a tree of denial does have some finitary expression. In my opinion, Wittgenstein's intentions are best approximated under an assumption that the general propositional form determines a direct denial relation the class of finitary formulas. I will adopt this assumption in what follows, noting that it may substantially constrain the class of acceptable implementations.

4 Formalizing truth-functionality

As I argued in §**??**[[1]], Wittgenstein advocates a project of analysis. The pursuit of analysis begins with antecedently acknowledged, apparently material entailments. The aim is of analysis is to render those entailments formal: to rewrite propositions so in such a way that all apparently material entailments turn out to be merely logical ones. An entailment between propositions is formalized when those propositions are rewritten in such a way that the entailment holds merely in virtue of the structure of the resulting propositional signs.

The general propositional form describes a system of signs which should be adequate to the formalization of all and only those entailments which are antecedently recognized. In the previous section, I reconstructed such a notational system. But it is not enough merely to speculate that propositions could be rewritten as signs of such a system. Such a rewriting would be adequate to analysis iff the pretheoretic entailments coincide with the formal entailments. Hence the adequacy of analysis depends on the concept of formal entailment. In other words: it remains to explain just what it for one proposition, merely through its expression in the system $\mathfrak{Z}$, to follow logically from some others.

A first step toward an answer is that $\mathfrak{Z}$ determines an relationship of direct denial. That is to say, there is on the class of sentences a relation $\prec$ such that from the structure of any two notations it easy to see whether $\prec$ relates them. To write some propositions p and q as A and NA is to write them in such a way that the truth of q is just the untruth of p. So when propositions are brought under the notational system prescribed by the general propositional form, then some of the truth-functionality of propositions gets formalized immediately.

Of course, the truth-functional relations formalized immediately by the direct denial relation induce further formalized truth-functionality as well: for example, three propositions written as $N(NA, NB)$, $N(NB, NC)$ and $N(NC, NA)$ are thereby represented so that the truth of any two of them requires the truth of the third. So there arises a question just which distributions of truth and falsehood are precluded merely by the relation $\prec$.

In §??4.1, I'll argue that part of the formal concept of truth-functionality is relatively unproblematic: namely, that in virtue of which propositions are truth-functions of elementary propositions. However, but not all logical interdependence is vertical: for example, a conjunction entails the disjunction of its conjuncts. So in §§??[[]] and ??[[]], I turn to question how logical dependence would have been rendered formal in general. Here, the *Tractatus* seems to contain two competing conceptions.[39] On one conception, which I'll call combinatorial, the fundamental form of truth-functionality is the effect of the truth-values of elementary propositions on the truth-values of all propositions. On the other conception, which I'll call affirmation-theoretic, truth-functional dependence is given through the iteration of discrete steps, analogous to those of the construction of sentences. I'll argue that the formality of logic favors the affirmation-theoretic conception over the combinatorial. But some mathematical results impede the development of the affirmation-theoretic conception in full generality. So as we'll eventually see, neither path evidently sustains the idea that necessity can be seen to flow from the nature of signs, not without significant compromise.

4.1 Truth-functional determination upward

As I've contended, Wittgenstein presents a scheme under which all internal relations ought to be symbolically realized. Among all internal relations he highlights a pivotal class: those relations in virtue of which propositions are "truth-functions of elementary

[39]Something like this phenomenon of apparent competing explanations is discussed in Proops Proops (2002).

propositions" (T5). Here I'll argue that the explication $\mathfrak{Z}$ of Wittgenstein's scheme fulfills this part of its purpose.

Underlying the scattered *Tractatus* presentation of the scheme is a fairly natural idea. By stipulation, elementary propositions are truth-functions of elementary propositions (T5b). Furthermore suppose that if everything directly denied by a nonelementary proposition is a truth-function of elementary propositions then it must be so too. Finally suppose that the relation of direct denial is well-founded, so that there is no infinite descending chain

$$A_1 \succ A_2 \succ \cdots .$$

Then it would follow by induction on $\prec$ that every proposition—or, to be precise, every proposition as presented in $\mathfrak{Z}$—is a truth-function of elementary propositions. Of course, it will remain to make plausible that truth-functional dependencies are in some sense shown to derive from the structure of signs.

As I've argued, the relation $\prec$ formalizes that a nonelementary propositional sign is the result of a truth-operation on the multiplicity of signs B such that $B \prec A$. It's this fact about direct denial which is supposed to underwrite the formal transmission of truth-functionality. But how? It is clear that Wittgenstein's own truth-tabular notation cannot suffice. For that notation depends on an enumeration of truth-possibilities for truth-arguments. But if the number of truth-arguments is infinite, then distributions of truth-value over the truth-arguments would be nondenumerable.

However, I'll now argue that we can make sense of the idea that propositions expressed as results of iterated direct denial do thereby exhibit their own nature as truth-functions of elementary propositions. Such truth-functionality amounts to the fact that each distribution of truth-values over the bases of a denial determines the truth-value of the result. I'll argue that the symbolism of denial generates, for each distribution of truth-values of its bases, a manifestation of the truth-value of the result in a pattern of symbols.

The idea is simple. For A to deny B is for A to disagree with the truth of B; and for A to disagree with the truth of B is for B's truth to require A's falsehood. So if A denies B, then B's truth requires A's falsehood, and this requirement is intrinsic to the formalized circumstance that A denies B. Consequently, direct denial formalizes this relationship:

$$\frac{+B}{-A}\ {\scriptstyle B \prec A}$$

Conversely, for A to be expressed as a joint denial is for A to disagree only with those B such that $B \prec A$. So the falsehood of all B such that $B \prec A$ is all it takes for A to be true. So direct denial formalizes this relationship too:

$$\frac{(-B : B \prec A)}{+A}$$

These rules suffice to verify that, for example, the analysis of a materialconditional formalizes the intended truth-function.

$$\dfrac{\dfrac{+q}{-\mathrm{N}(q,\mathrm{N}(p))}}{+\mathrm{N}(\mathrm{N}(q,\mathrm{N}(p)))}\;;\;\dfrac{\dfrac{\dfrac{-p}{+\mathrm{N}(p)}}{-\mathrm{N}(q,\mathrm{N}(p))}}{+\mathrm{N}(\mathrm{N}(q,\mathrm{N}(p)))}\;;\;\dfrac{\dfrac{-q \qquad \dfrac{+p}{-\mathrm{N}(p)}}{+\mathrm{N}(q,\mathrm{N}(p))}}{-\mathrm{N}(\mathrm{N}(q,\mathrm{N}(p)))}$$

Let's say that a truth-possibility is a distribution of truth and falsehood over elementary propositions. And, say that B is a *truth-argument* of A iff B is an elementary proposition which is a $\prec$-descendant of A. The principles [[]] together ensure for each truth-possibility Π for the truth-arguments of A, there is a pattern of symbols to witness that Π determines the truth-value of A. This follows merely from the assumption that $\prec$ is wellfounded on $\mathfrak{Z}$.

But first a bit more notation. Let's use lowercase Greek letters $\phi, \psi, \ldots$ for signed formulas $+A, -A, +B, \ldots$; and let's use uppercase Greek letters Γ, Δ, Π for collections of signed formulas. Finally let's write $\Gamma \Vdash \phi$ to mean $(-\mathrm{N}i), (+\mathrm{N}i)$ generate a pattern of symbols to witness that Γ requires ϕ.

Proposition 1. *Let Π be a truth-possibility for the truth-arguments of A. Then $\Pi \Vdash +A$ iff $\Pi \nVdash -A$.*

Proof. Since $\prec$ is wellfounded we can argue by induction. In the right-to-left direction, suppose that $\Pi \nVdash +A$. Then there must be a $B \prec A$ such that $\Pi \nVdash -B$ since otherwise we'd have $\Pi \Vdash +A$ by $(+\mathrm{N}i)$. By induction hypothesis, it follows that $\Pi \Vdash +B$, and so $\Pi \Vdash -A$ by $(-\mathrm{N}i)$.

Conversely, suppose toward a contradiction that $\Pi \Vdash +A$ while $\Pi \Vdash -A$. If A were elementary, then the hypothesis implies that both $+A$ and $-A$ are in Π, contradicting that Π is a truth-possibility. So suppose A is nonelementary, and assume the induction hypothesis. Since Π contains only elementary propositions, therefore $\Pi \Vdash +A$ only if $\Pi \Vdash -B$ for all $B \prec A$. And similarly $\Pi \Vdash -A$ only if $\Pi \Vdash +B$ for some $B \prec A$. This contradicts the induction hypothesis. □

4.2 Truth-functional determination generally

We've now seen an account of how it may come about, merely in virtue of the structure of signs, that propositions are truth-functions of elementary propositions. But this does not account for the showing of all truth-functional dependence. For example, it explains that a proposition's falsehood requires the truth of its negation. But it doesn't explain the converse. In this section, I'll propose a couple of ways to extend the previous construction. The goal will be to clarify the possibilities for claiming that all logical relationships might be formalized along lines developed in the *Tractatus*.

4.2.1 Following as truth-ground inclusion

Let's begin by recalling the last sentence of 6.124: "if we know the syntax of any one sign-language, then all the propositions of logic are already given." How might that be so? Recall that when a proposition is expressed in the sign A, then its truth-arguments are the leaves of its tree of denial. We've seen that each formalized proposition is a truth-function of its truth-arguments. A truth-ground of A is a truth-possibility for the truth-arguments of A in accordance with which A must be true (5.1). The *truth-condition* of a proposition A is the specification of its truth-grounds (4.431).

Let's now consider some logical relationships which are so far unexplained:

> Among the possible groups of truth-conditions there are two extreme cases.
>
> In the one case the proposition is true for all the truth-possibilities of the elementary propositions. We say that the truth-conditions are *tautological*.
>
> In the second case the proposition is false for all the truth-possibilities. The truth-conditions are *self-contradictory*. [4.46a-c]

Thus, a tautology is a propositions whose truth is required not just by some, but by all truth-possibilities for its truth-arguments. From this characterization, Wittgenstein infers that

> A tautology has no truth-conditions, since it is unconditionally true: and a contradiction is true on no condition. [4.46e]

Wittgenstein's reasoning here is rather quick. Consider a simple case first. Suppose that ϕ is required both by $+P$ and by $-P$. Then, he concludes, ϕ is simply required. For example, it is routine to verify, using ($+\mathrm{N}i$) and ($-\mathrm{N}i$), that $+\mathrm{N}(\mathrm{N}(P,\mathrm{N}(P)))$ is required both by $+P$ and by $-P$. Hence $+\mathrm{N}(\mathrm{N}(P,\mathrm{N}(P)))$ would be simply required. Or in other words, $\mathrm{N}(\mathrm{N}(P,\mathrm{N}(P)))$ is tautological. The crucial assumption of this argument could be represented like this:

$$\dfrac{\begin{array}{c}[+P]\\ \vdots\\ \phi\end{array}\qquad\begin{array}{c}[-P]\\ \vdots\\ \phi\end{array}}{\phi}$$

It is clear that this assumption doesn't disrupt the goal of rendering truth-functionality somehow manifest. The above scheme can be filled out by witnesses that ϕ follows both from $+P$ and from $-P$. The result is a pattern of symbols which arguably witnesses the the fact that its root formula is a tautology.

But Wittgenstein's intention would seem to be more general than that. Consider for example the sign $\mathrm{N}(\overline{x\mathrm{N}(Fx,\mathrm{N}(Fx))})$. This is the joint denial of all propositions $\mathrm{N}(Fa,\mathrm{N}(Fa))$ for a a name. So its truth-arguments are precisely those propositions Fa. Now consider an arbitrary truth-possibility Π for these truth-arguments. For each a we must have either $+Fa$ or $-Fa$ in Π. Both $+Fa$ and $-Fa$ require $-\mathrm{N}(Fa,\mathrm{N}(Fa))$, so this is required by Π itself. Since a was arbitrary, we have seen that Π requires $-\mathrm{N}(Fa,\mathrm{N}(Fa))$ for all a. But that's just to say that Π requires $+\mathrm{N}(\overline{x\mathrm{N}(Fx,\mathrm{N}(Fx))})$. Now in turn, Π was an arbitrary truth-possibility. So every truth-possibility requires $+\mathrm{N}(\overline{x\mathrm{N}(Fx,\mathrm{N}(Fx))})$. It follows that the truth of $\mathrm{N}(\overline{x\mathrm{N}(Fx,\mathrm{N}(Fx))})$ is required unconditionally, making it a tautology.

Again this argument goes beyond mere introduction rules (in the concluding "it follows that".) Now if the number of propositions Fa is finite, then this argument can be justified by repeated use of the assumption just introduced. And again the tautologousness of its root will be manifest by a pattern of symbols. But, not otherwise. Rather, Wittgenstein's characterization of tautologousness would seem to require a principle that's more general.

Where Γ is a collection of unsigned formulas, let's write $|\Gamma|$ for the collection of unsigned formulas underlying the elements of Γ. Let's write $\mathfrak{E}_\Gamma$ for those elementary propositions in the $\prec$-ancestry of Γ. Then the new principle can be stated like this:

$$\frac{\left(\begin{array}{lcl} [\Pi] & & \\ \ \vdots & : & |\Pi| = \mathfrak{E}_\phi \\ \ \phi & & \end{array}\right)}{\phi}$$

In words, the idea is that a witness for ϕ is given by witnesses that ϕ follows from each truth-possibility Π for the truth-arguments of ϕ.

At this point we can explicate the general concept of tautology in the *Tractatus*. The rule together with $(-\mathrm{N}i)$ and $(+\mathrm{N}i)$, determines a collection of propositional signs of $\mathfrak{Z}$ whose truth is unconditionally required. To be a tautology is to be one of these signs. As befits its introduction in the *Tractatus*, the concept of tautology is of a piece with that of contradiction. A contradiction is a sign whose falsehood is required by the same system of rules.

But as we've seen in §1[[ref]], Wittgenstein holds that tautology and contradiction are limiting cases, even just outgrowths of the symbolism like a zero of arithmetic. Does logic have anything to say about the important case? Do real propositions, those with susceptibility both to truth and to falsehood, have any interesting logical features? Of course!

> If the truth-grounds which are common to a number of propositions are all also truth-grounds of some one proposition, we say that the truth of this proposition follows from the truth of those propositions. [5.11]
>
> In particular the truth of a proposition p follows from that of a proposition q, if all the truth-grounds of the second are truth-grounds of the first. [5.12]

Thus, following is truth-ground inclusion.

For example, the only truth-ground of $\mathrm{N}(\mathrm{N}(P), \mathrm{N}(Q))$ is $+P, +Q$. This is certainly among the truth-grounds of $\mathrm{N}(\mathrm{N}(P,Q))$. We wish to conclude that $+\mathrm{N}(\mathrm{N}(P), \mathrm{N}(Q))$ requires $+\mathrm{N}(\mathrm{N}(P,Q))$. But it doesn't follow merely from the principles developed so far. The same goes even for the claim that $+NP$ requires $-P$. One way to see this is to note that on the rules developed so far, if $|\phi|$ is not a subformula of $|\psi|$, then there is a witness that ψ follows from Π, ϕ only if there is also a witness that ψ follows from Π.

Essentially, what's required is this:

if for every truth-possibility Π for the truth-arguments of $|\Gamma|$ and $|\phi|$
there is a witness that some ψ amongst Γ^*, ϕ follows from Γ, Π,
then, there is a witness that ϕ follows from Γ.

There are several ways to secure this. But let's just wire it up directly. Write

$$\begin{array}{c}\Gamma\\ \vdots\\ (\Delta)\end{array}$$

for a witness that some ϕ or other in follows from Γ, with ϕ amongst Δ (understanding this diagram quantifier to take narrowest scope). Also write Γ^* for those ψ^* with ψ in Γ. Then a principle which suffices, together with introduction rules, to axiomatize Wittgenstein's notion of following can be depicted

$$\frac{\Gamma, \left(\begin{array}{c}[\Pi]\\ \vdots\\ (\Gamma^*, \phi)\end{array} \quad : \quad |\Pi| = \mathfrak{E}_{\Gamma,\phi} \right)}{\phi}$$

For example, a witness that $-P$ follows from $+NP$ would accordingly look like this:

$$(\vDash)\ \frac{\dfrac{[+P]}{-\mathrm{N}(P)} \qquad [-P] \qquad +\mathrm{N}(P)}{-P}$$

In this subsubsection, we saw that although the introduction rules ($-\mathrm{N}i$) and ($+\mathrm{N}i$) yield manifestations that propositions are truth-functions of elementary propositions, they do not suffice to show that a proposition is a tautology. But it was enough to add a kind of infinitary principle of bivalence: if something is required by any distribution of truth-value over elementary propositions, then it is required absolutely. In turn, the concept of following required further generalization. Thus we reached the principle ($\vDash$). In sum, ϕ can be said to follow from Γ if the principles ($+\mathrm{N}i$), ($-\mathrm{N}i$), and ($\vDash$) together generate a witness that ϕ follows from Γ. The form of this analysis is important: a conclusion follows from some premises because something shows that it does.

4.2.2 Worries about truth-ground inclusion

We've just considered a concept of following. This concept may be a good candidate for what Wittgenstein understood as the ground of logical connectedness. Indeed, it's a pretty straightforward elaboration of what Wittgenstein actually says. But, there are some reasons to worry that the account of following as truth-ground inclusion undermines what I've argued is a broader philosophical aim of the *Tractatus*, namely to show that logical relationships are symbolically realized.

To begin with, let's consider a piece of not-even-science fiction. Say that a lineup is orderly if exactly one person in the line is not behind anybody, everybody else in the line is immediately behind exactly one person, and nobody has two people behind them. Say that a world is orderly if everybody stands in one and the same orderly line. On the other hand, let's say that an object is properly contemplated if it is the only thing that somebody or other is thinking about. Suppose that a god (or the TSA) has brought about an orderly world. Could that god also bring about that every object whatsoever is properly contemplated? Or has this question been answered already, in the negative? It should be clear that orderliness and proper contemplation can each be expressed by a single propositions; let's call them ORDERLINESS and CONTEMPLATION. Now, the question is a purely logical one: does ORDERLINESS preclude CONTEMPLATION?

The answer to this question of logic depends on a comparison of cardinality: specifically, are there as many people as there are objects? In turn, comparisons of cardinality would seem to depend on the existence of functions (or in Russellian: many-one relations). Does there exist a function which maps the people onto the objects? If no such function exists, then the actions of a god or the TSA which bring about an orderly world automatically settle, in the negative, whether everything is properly contemplated. But not otherwise.

But how are we to understand the relevant notion of existence of a function? An obvious proposal is to explain the existence of a surjective function through the obtaining of certain general facts. Let PEOPLEONTOUNIVERSE(R) be the proposition

$$\forall x(\text{PERSON}(x) \to \exists! y Rxy) \land \forall y \exists x(\text{PERSON}(x) \land Rxy).$$

Then, for there to be a function from one class $\text{PEOPLE}^{\mathscr{W}}$ onto another class UNIVERSE would be for it to happen to be the case that the actual world satisfies

$$\mathscr{W} \text{ makes this true: PEOPLEONTOUNIVERSE}(R) \tag{1}$$

for some R or other. From Wittgenstein's point of view, this is an obvious violation of the determinacy of sense. It makes the purely logical question of compatibility of ORDERLINESS and CONTEMPLATION depend on what happens to be the case in this or that world $\mathscr{W}$. For example, the actual world might so happen to falsify (1). We would then be assured that the actual world also satisfies

$$\mathscr{W} \text{ makes this true: ORDERLINESS} \to \neg\text{CONTEMPLATION}. \tag{2}$$

Nonetheless, some other world might fail to satisfy (1), and all bets would be off as to whether it satisfies (2). Wittgenstein ought to object to this explanation that according to it, ORDERLINESS does not represent "off its own bat"; the question of what it implies depends on what happens to be the case.

Is there a way to explain comparisons of cardinality while respecting determinacy of sense? A natural idea is to appeal not to actual existence of surjections, but to their mere possibility. Roughly, one might say that $\text{PEOPLE}^{\mathscr{W}}$ is at least as large as UNIVERSE just in case some such function does exist in a world which agrees with $\mathscr{W}$ on PEOPLE. More precisely, say that $\text{Diagram}_{\mathscr{W}}(\text{PEOPLE})$ is the class of all propositions true in $\mathscr{W}$ to the effect that this or that thing is or isn't a person. Now, the proposal is that $\textsc{People}^{\mathscr{W}}$ is at least as large as $\textsc{Universe}$ provided that

$$\text{Diagram}_{\mathscr{W}}(\textsc{People}) \cup \{\textsc{PeopleOntoUniverse}(R)\}$$

is consistent.

Now, it would seem that this modal explanation of cardinality must be extensionally correct, if the system of logical relationships is to be coherent at all. The modal explanation also does not depend on what happens to be the case in any particular world as opposed to another. So it does not appear to violate the determinacy of sense. Rather, the problem is that it is obviously circular. After all, the point of making sense of cardinality here was to explain what makes it possible for a Wittgensteinian god to do something, or in other words what would make some propositions consistent.

I suggest, then, that from Wittgenstein's point of view, logical relationships cannot be explained by appeal to some prior purchase on the relationships of cardinality. For in turn, relationships of cardinality depend on the existence and nonexistence of functions; yet Wittgenstein has no room for the relevant notion of existence and nonexistence of functions except as a redescription of relationships of logic. Logic explains cardinality, rather than vice versa.

Let's now consider what happens to cardinality, if truth-ground inclusion explains logic? The question whether ORDERLINESS precludes CONTEMPLATION becomes the question whether there exists a pattern of symbols

$$\frac{+\textsc{Orderliness},\ \left(\begin{array}{c}[\Pi]\\ \vdots\\ (-\textsc{Orderliness},-\textsc{Contemplation})\end{array} \quad : \quad |\Pi| = \mathfrak{E}_{O,C}\right)}{-\textsc{Contemplation}}$$

where $\mathfrak{E}_{O,C}$ is the collection of truth-arguments of ORDERLINESS, CONTEMPLATION.

Now in order genuinely to witness the preclusion of CONTEMPLATION from FOLLOWING, an array of symbols must witness that

$$\Pi \Vdash -\textsc{Orderliness}, -\textsc{Contemplation}$$

for every truth-possibility Π for $\mathfrak{E}_{O,C}$. It's not clear, however, that whether the array covers every truth-possibility can be inspected from the array itself. Rather, at least the in case of infinitely many truth-arguments, the required exhaustiveness would seem irreducibily to be a relationship between the array and the class of all truth-possibilities.

For example, consider Wittgenstein's own formulation of the independence of elementary propositions.

> The world divides into facts. [1.2]
>
> Any one can either be the case or not be the case, and everything else remain the same. [1.21]

Suppose that Π be the pattern of obtaining and nonobtaining of elementary propositions which is determined by the remark 1.2. And now let $\mathscr{F}$ be smallest class of truth-possibilities with Π in $\mathscr{F}$, such that $\mathscr{F}$ contains every truth-possibility differing over at most one elementary proposition from any truth-possibility it contains. Then $\mathscr{F}$ satisfies both 1.2 and 1.21. Yet surely, it doesn't fulfil Wittgenstein's intention. For example, suppose, for example, that some existential generalization $\mathscr{A}$ has infinitely many instances which are true at some element of $\mathscr{F}$. Then $\mathscr{A}$ is true at every element in $\mathscr{F}$. Therefore, any existential generalization with infinitely many true instances would be a tautology.

Of course, the class of all truth-possibilities might be populated a little more richly than Wittgenstein's own formulations require. But it's not clear that the question what truth-possibilities exist must be fixed merely by what makes anything into propositional symbols. To bring this out, it might be helpful to consider a seemingly parallel issue: the logical determination of the totality of objects. Here Wittgenstein was somewhat more explicit:

> The world is everything that is the case. [1]
>
> The world is the totality of facts, not of things. [1.1]
>
> The world is determined by the facts, and by these being all the facts. [1.11]
>
> For the totality of facts determines both what is the case, and also all that is not the case. [1.12]

Thus, if $Fa, Fb, \ldots$ are the totality of values of Fx, then a distribution $-Fa, -Fb, \ldots$ must by itself determine that no instance of Fx is true. A similar acknowledgment appears in a quite different context:

> The "experience" which we need to understand logic is not that such and such is the case, but that something *is*; but that is *no* experience.
>
> Logic *precedes* every experience—that something is *so*.
>
> It is before the How, not before the What. 5.552a-c

Presumably, here the "What" which is required for logic to get going includes the determination of which objects there are.

Clearly Wittgenstein holds that the truth of a universal generalization follows given the truth of sufficiently many of its instances. Thus, it might be wondered: the rule for concluding the truth of $NxFx$ requires only an array of all values of Fx. But presented with an array of such values, can't I still worry that some other instance Fb has been overlooked? Why can't we assume Wittgenstein to take for granted the determinate totality of truth-possibilities in just the way that he assumes as determinate the universe of objects?

But the issue at stake here is the nature of logical relationships, which are supposed to be symbolically realized. With respect to this issue, the two requirements of determinateness are not on a par. For there are a couple of reasons for thinking that the origin of propositional symbols already requires the determination of the totality of objects. First, generalized propositions are truth-functions of a class of all elementary propositions with a common form. Therefore, for something to be any single general proposition already requires that a significant portion of the totality of elementary propositions have been fixed. Second, it is obvious that nothing can count as the totality of propositions except by reference to the totality of elementary propositions and hence also to the totality of objects. The class of all truth-possibilities is not comparably entangled in the nature and identity of propositions. For the system of all propositions ought to be constructed from the system of names and of elementary propositions, by purely finitary means. Discriminating between different ways the class of truth-possibilities could be populated goes way beyond the resources required for the building of all propositions.

The problem for the truth-ground analysis which was posed in this section can be summarized like this. Truth-ground inclusion varies with the existence and nonexistence of arbitrary functions on the domain of all objects. Equivalently, it depends on how the class of truth-possibilities for elementary propositions has been populated. So it cannot be fixed merely by what makes anything into propositional symbols. So, if truth-ground inclusion explains logic, then logic isn't symbolically realized.

4.2.3 A affirmation-theoretic approach

To summarize, then, the *Tractatus* explains how it should come about, in virtue of the structure of signs, that propositions are truth-functions of elementary propositions. But, this does not by itself suffice to explain how all truth-functional interaction could be manifested in signs. For example, it explains that a proposition's falsehood yields the truth of its negation, but does not explain the converse. The attempt to restore this symmetry by appeal to the notion of truth-ground inclusion seems to help for those signs which depend for their truth on finitely many elementary propositions. Even completely trivial examples, like the fact that the truth of a universal generalization requires the truth of its instances, cannot be exhibited except by laying out the entire collection of truth-combinations. At this point, the situation looks rather dire.

I think a little more can be done on behalf of Wittgenstein here. First of all, note that there is a somewhat more obvious objection to the analysis of following as truth-ground inclusion. That analysis presupposes certain fundamental constraints on the distribution of truth and falsehood. For example, it assumes that if A denies B then B is false only if A is true. Why shouldn't what's really fundamental be constraints of the sort that the truth-ground analysis takes for granted? What's so special about the exorbitantly commital principle ($\models$)?

Indeed, there are textual reasons to suppose Wittgenstein had some independent reason for skepticism about the truth-ground analysis.

> If a god creates a world in which certain propositions are true, he creates thereby a world in which all propositions consequent on them [*Folgesätze*] are true. And similarly he could not create a world in which the proposition "p" is true without creating all its objects. [5.123]

The point of this remark seems to be that once the truth of some propositions has been secured, no further task remains to bring about what is consequent upon them. Suppose, however, that the truth-ground analysis is applied to the notion of consequence in play here. Then, the remark becomes tautologous, and its point is lost. Rhetorically, 5.123 destabilizes the truth-ground analysis given at 5.11-5.12.

For a proponent of the truth-ground analysis the next remark should be even more puzzling:

> A proposition affirms [*bejaht*] every proposition which follows from it. [5.124; Ogden-Ramsey have "asserts"]

The word translated here as "affirmation", namely *Bejahung*, literally means "saying yes to something". It partners naturally with *Verneinung*—which can be translated either (following Ogden-Ramsey) as "denial" or (following Pears McGuinness) as "negation". But Wittgenstein's usage is pointedly strange: contrary to that in Frege 1918, for example, affirmation and denial are not, for Wittgenstein, something done by us; rather it is done by propositions.[40] It's plausible to suppose that in this talk of propositions, rather than of people, as making affirmations and denials, Wittgenstein develops the theme that gets summarized in 6.124's declaration that in logic it's not people but signs that speak. That is, affirmation and denial are just what happens in logic. About affirmation and denial Wittgenstein does dilate a little:

> "$p.q$" is one of the propositions which [affirms] "p" and at the same time one of the propositions which [affirm] "q".
>
> Two propositions are opposed to one another if there is no significant proposition which [affirms] them both.
>
> Every proposition which contradicts another, denies it. [5.1241]

I've urged that the conception of following as truth-ground inclusion enjoys no obvious grounding in the structure of signs. In contrast, Wittgenstein's hints about affirmation generally indicate an essentially syntactical origin. A paradigm would be that the conjunction of a proposition with another affirms each of them. How? It affirms them because it shows that it does, in the way it's written: as their conjunction.

Because affirmation is so rooted in how propositions are realized in signs, the coincidence of affirmation and following declared at 5.124 implies what's remarked at 5.13:

> That the truth of one proposition follows from the truth of other propositions, we perceive from the structure of the propositions. 5.13

So, 5.12 through 5.13 form a continuous line of argument. First, 5.12 records the observation that consequence corresponds to truth-ground inclusion. But then, 5.123 debunks the suggestion that truth-ground inclusion could explain the nature of logical connectedness. Logical connections are just what make something a truth-ground! Rather, he concludes at 5.124, consequence originates in relationships of affirmation and denial between signs which obtain in virtue of their character merely as signs.

So affirmation and denial by signs is the means of apprehension of necessity. But in turn, affirmation and denial make their appearance in signs thanks to an activity described in a passage I quoted much earlier:

[40] Shieh (forthcoming) develops an account of logical necessity rooted in patterns of norms of affirmation and denial, and I'm indebted to Shieh's appeal to the notion of pattern. However, since affirmation is done not by people but by propositions, it's not quite right to say that affirmation is governed by norms. Rather, by Shieh's lights, it might be better to say that patterns of affirmation and denial by propositions are—at least if we step outside of logic!—patterns of norms.

> Propositions stand to one another in internal relations.
>
> We can bring out these internal relations in our manner of expression, by presenting a proposition as the result of an operation which produces it from other propositions (the bases of the operation). [5.2]

Any symbolism is woven together by syntactical hooks and manipulations (5.511) whereby its use acknowledges propositions to stand to each other in internal relations. To write a proposition as the direct denial of some others presents it as that whose truth just is their falsehood, and also conversely.

We are now in a position to see why it was so important to Wittgenstein that propositions be written as the results of applying the operator N. In this notation, each nonelementary proposition is presented as something it logically does: deny some other stuff. It remains to try again at spelling out how, once propositions are so presented, all their logical interconnectedness could be made manifest in patterns of signs. I will draw inspiration from what may later have seemed to Ramsey like one of the least auspicious remarks in the entire book.

> If a notation is fixed, there is in it a rule according to which all the propositions denying *p* are constructed, a rule according to which all the propositions asserting *p* are constructed, a rule according to which all the propositions asserting *p* or *q* are constructed, and so on. These rules are equivalent to the symbols and in them their sense is mirrored. [5.514]

Recall that the operator N is a means of constructing new signs from signs constructed already. Accordingly,

- the truth of A is the falsehood of all $B \prec A$;
- the falsehood of A is the truth of some $B \prec A$.

The method of construction is the source of logical connectedness. That is, as was urged in §[[ref]], the logical connectedness of propositions derives from the way in which a proposition pictures through others, from how its specification of its truth-condition may be parasitic on theirs.

Accordingly, half of the ensuing connections have been formulated already:

- $\dfrac{+B}{-A}$, for any $B \prec A$
- $\dfrac{(-B : B \prec A)}{+A}$

To finish the job, a natural strategy is supplement these "introduction rules" with corresponding "elimination rules":

- $\dfrac{+A}{-B}$, for any $B \prec A$

- $$\dfrac{-A, \quad \begin{pmatrix} [+B] & & \\ \vdots & : & B \prec A \\ \phi & & \end{pmatrix}}{\phi}$$

However, this doesn't quite work. Just adding elimination rules won't underwrite demonstration that anything is a tautology. Indeed, it's not clear that elimination rules ought to suffice. Elimination of a connective should not yield a conclusion not required for its introduction. But, some acknowledgments inhere purely in the use of + and —: like, perhaps, that nothing isn't required by something's simultaneous truth and falsehood.

Now, one solution is simply to supplement introduction and elimination rules with something further. But together with introduction rules, elimination rules can even be obviated by a natural principle governing relationship between + and —, i.e., truth and falsehood. This is the so-called "Smileian Reductio":[41]

$$(☺)\ \dfrac{\begin{matrix}[\phi]\\ \vdots \\ \psi\end{matrix} \qquad \begin{matrix}[\phi]\\ \vdots\\ \psi^*\end{matrix}}{\phi^*}$$

Together with (+Ni) and (—Ni), Smileian reductio yields the elimination patterns, like this:

$$\dfrac{\dfrac{\dfrac{[+B]_1}{-A} \qquad +A}{-B} \qquad [-B]_1}{-B}\,{}_1$$

$$\dfrac{\dfrac{\begin{pmatrix} [+B]_1 & & \\ \vdots & : & B \prec A \\ \dfrac{\phi \qquad [\phi^*]_2}{-B} & & \end{pmatrix}}{+A} \qquad -A}{\phi}\,{}_2$$

Just to emphasize, I doubt that there's a reason, at least in this context, to suppose that the Smileian essentially warrants pride of place over various alternatives.[42] Indeed, as emphasized in §2[[ref]], even the choice of language $\mathfrak{Z}$ is arbitrary already.

Let's now write that $\Gamma \Vdash \phi$ when the patterns (+Ni), (—Ni), and (☺) can be combined to demonstrate that Γ requires ϕ. I propose that the relation $\Vdash$ can be applied interpretively in the following way: that if $\Gamma \Vdash \phi$, then it holds just by the

[41][[cite]] Rumfitt.

[42]For example, the method of semantic tableaux developed in Rogers and Wehmeier **?** could be adapted to this setting as well.

nature of those symbols that Γ requires ϕ. Specifically, Wittgenstein talks about a proposition A as "affirming" B: I propose a sufficient condition for this to be that $+A \Vdash +B$. Likewise, A would deny B when $+A \Vdash -B$. And conversely, A agrees (or disagrees) with Γ when $\Gamma \Vdash +A$ (or $\Gamma \Vdash -A$).

I will now argue that under some conditions, the relation $\Vdash$ actually coincides with truth-ground inclusion. Thus, $\Vdash$ substantiates the concept of affirmation as a relationship of logical truth-dependence which is grounded in the structure of signs. On this analysis, some such dependences would be acknowledged by the writing of a proposition as the result of a truth-operation; given the initial acknowledgments, affirmation and denial would then ramify through the totality of propositional signs according to transparent syntactical patterns.

Recall the remark 5.124, which introduces the concept of affirming by propositions with the claim that a proposition affirms what follows from it.[43] Since this remark occurs on the heels of the truth-ground inclusion analysis of following, therefore "follows" in 5.124 should be understood to express truth-ground inclusion. Thus, 5.124 would amount to the claim that truth-ground inclusion relationships coincide with those logical requirements which can be found to obtain thanks to the symbolizations of their relata.

In §**??**[ ]], we found an analysis that Π is a truth-ground of A. Let's now generalize the idea of the proposal to signed formulas:

- $\Pi \vdash \phi$ iff the rules $(+\mathrm{N}i), (-\mathrm{N}i)$ suffice to demonstrate that Π requires ϕ.

We can now adapt the concept of truth-ground inclusion to signed formulas, thus:

- $\Gamma \vDash \phi$ iff for every truth-possibility Π for the truth-arguments of Γ, ϕ, if $\Pi \vdash \psi$ for all ψ in Γ, then also $\Pi \vdash \phi$.

Note that by definition, if $\Gamma \vdash \phi$ then $\Gamma \Vdash \phi$ and $\Gamma \vDash \phi$.

I will now argue that under some conditions, $\Gamma \Vdash \phi$ iff $\Gamma \vDash \phi$. From this, together with 5.124, it will follow that A affirms B iff $+A \Vdash +B$. So under those conditions, $\Vdash$ would yield an extensionally adequate analysis of affirmation, and more generally of the realization of logical dependence in signs as this can be understood in the *Tractatus*. [[Moreover, it is at least arguable that $\Vdash$ does in virtue of the nature of signs.]]

In one direction, the claim is straightforward.

Proposition 2. *If* $\Gamma \Vdash \phi$, *then* $\Gamma \vDash \phi$.

Proof. Suppose that $\Gamma \Vdash \phi$. Let Π be a truth-possibility for the truth-arguments of Γ, ϕ such that $\Pi \vdash \psi$ for all ψ in Γ. It suffices to show that $\Pi \vdash \phi$. The definitions of $\Vdash$ and $\vdash$ imply that $\Pi \Vdash \psi$ for all ψ in Γ. Given a witness of $\Gamma \Vdash \phi$, the result of replacing each of its ψ-leaves with a witness of $\Pi \Vdash \psi$ is in turn a witness of $\Pi \Vdash \phi$. This will

[43] At 4.064 Wittgenstein had already complained, evidently against Frege (1879), that a proposition isn't given a sense by affirmation; but instead the proposition affirms its own sense and denies the sense of others.

fail to witness $\Pi \vdash \phi$ only by containing uses of (☺), which is to say parts of the form

$$\frac{\begin{array}{c}\Pi,[\theta]\\ \vdots\\ \eta\end{array}\qquad\qquad\begin{array}{c}\Pi,[\theta]\\ \vdots\\ \eta^*\end{array}}{\theta^*}\ .$$

It will suffice to show that every such part can be replaced by a witness of $\Pi \vdash \theta^*$. By induction, the component witnesses of $\Pi, \theta \Vdash \eta$ and $\Pi, \theta \Vdash \eta^*$ can be assumed to contain no uses of (☺). So actually $\Pi, \theta \vdash \eta$ and $\Pi, \theta \vdash \eta^*$. Now, Proposition **??**[]] says that $\Pi \vdash \eta$ only if $\Pi \nvdash \eta^*$. But if $\Pi \vdash \theta$, then $\Pi \vdash \eta$ while $\Pi \vdash \eta^*$, a contradiction. Proposition **??**[[]] also says that $\Pi \vdash \theta^*$ if $\Pi \nvdash \theta$. Therefore $\Pi \vdash \theta^*$, as desired. □

The converse question is whether truth-ground inclusion implies affirmation. As we'll see, this turns on matters of cardinality. The finite case is easy.

Proposition 3. *Suppose that* Γ, ϕ *together have finitely many truth-arguments. Then* $\Gamma \vDash \phi$ *only if* $\Gamma \Vdash \phi$.

Proof. Suppose that $\Gamma \vDash \phi$. Now, let $\Pi = \pi_1, \ldots, \pi_k$ be a truth-possibility for the truth-arguments of Γ, ϕ. By the definition of $\vDash$, either $\Pi \vdash \phi$ or $\Pi \vdash \psi^*$ for some ψ^* in Γ. If $\Pi \vdash \psi^*$, then a use of (☺) shows that $\Pi, \Gamma \Vdash \phi$. So $\Pi, \Gamma \Vdash \phi$ for all Π.

Three uses of (☺) establish a form of excluded middle: that if $\theta, \Delta \Vdash \eta$ and $\theta^*, \Delta \Vdash \eta$ then $\Delta \vdash \eta$. Note that a form of excluded middle is derivable:

$$\frac{\dfrac{\begin{array}{c}\Gamma,[\theta]_1\\ \vdots\\ \phi\end{array}\qquad [\phi^*]_3}{\theta^*}\,{}_1\qquad\qquad \dfrac{\begin{array}{c}\Gamma,[\theta^*]_2\\ \vdots\\ \phi\end{array}\qquad [\phi^*]_3}{\theta}\,{}_2}{\phi}\,{}_3\ .$$

By induction on the π_i it follows that $\Gamma \vdash \phi$. □

The countable case is just a bit more delicate. To simplify the argument, let's write $\Gamma \Vdash$ to mean that $\Gamma \Vdash \phi^*$ for some ϕ in Γ. And write $\Gamma \vDash$ to mean that $\Pi \nvdash \Gamma$ for all truth-possibilities Π for the truth-arguments of Γ. The rule of (☺) implies that the following result is enough.[44]

Proposition 4. *Suppose that the ancestry of* Γ *is countable. Then* $\Gamma \vDash$ *only if* $\Gamma \Vdash$.

Proof. Let's try to generate from Γ a tree of sets of signed formulas, which will yield either a counterexample to $\Gamma \vDash$ or a witness of $\Gamma \Vdash$.

Let Γ be the root. Say that a node is conflicted if it contains both some signed formula and its opposite. Take the conflicted nodes to have no children. But consider

[44] Actually, that's the only use of (☺) required for completeness. The argument below uses only the introduction and elimination rules.

Δ unconflicted. Say that the rank of a node is the number of nodes which are its ancestors in the tree. If Δ has even rank, then let its one child be the result of adding to Δ each formula $-B$ such that $B \prec A$ for some A with $+A$ in Δ. On the other hand, we assumed that the $\prec$-ancestry of Γ is countable; let those formulas be uniquely indexed by finite ordinals. And suppose that Δ has odd rank. In that case, let Σ be a child of Δ iff $\Delta \subseteq \Sigma$ and for each A in Δ with index less than the rank of Δ, there's a $+B$ in Σ with $B \prec A$.

Suppose that some branch of the tree does not end in a conflict. Let Δ be the union of its nodes, and let Π be the collection of signed elementary propositions in Δ. By construction, Π must be a truth-possibility for the truth-arguments of Γ. Let's now argue, by induction on $\prec$, that $\Pi \vdash \phi$ for each ϕ in Γ. It suffices to consider ϕ nonatomic. Suppose that ϕ is $+A$. Then $-B$ is in Γ for each $B \prec A$, and $\Pi \vdash -B$ by induction hypothesis. So ($+$Ni) gives $\Pi \vdash +A$.. Alternatively, suppose that ϕ is $-A$. Then some $+B$ is in Γ for some $B \prec A$, and $\Pi \vdash +B$ by induction hypothesis, so that ($-$Ni) gives $\Pi \vdash -A$. Therefore, $\Pi \vdash \phi$ for all ϕ in Γ. So $\Gamma \nvDash$.

On the other hand, suppose that every branch ends in a conflict. Let Δ be an arbitrary node. Then no infinite branch descends from Δ. By induction, we may therefore suppose that $\Sigma \Vdash$ for every child Σ of Δ. If the rank of Δ is even, then ($+$Ne) implies that $\Delta \Vdash \phi$ for all elements ϕ of Δ's one child. Replacing each ϕ-leaf in a witness that $\Sigma \vDash$ with a witness of $\Delta \vDash \phi$ gives a witness of $\Delta \Vdash$. Finally suppose that Δ has odd rank. Then there are some $-A_1, \ldots, -A_k$ in Δ such that $\Delta, +B_1, \ldots, +B_k \Vdash$ for all $B_1, \ldots, B_k$ with $B_1 \prec A_1, \ldots, B_k \prec A_k$. So, k applications of ($-$Ne) yield $\Delta \Vdash$. Since Γ is itself a node, therefore $\Gamma \Vdash$. □

On the reading advanced in §1, Wittgenstein advocates a shift in the aims of logic, from the recognition of truths to the clarification of possibilities. Such clarification is the activity of analysis, giving signs to propositions so that necessities and possibilities would then derive from patterns of symbolism. But could there be patterns of symbolism in virtue of which necessities may be said to have been formalized? Suppose that the number of objects is countable. Then, I propose, the last few results show that the answer is plausibly yes.

But the assumption of countability cannot be dropped. That is to say, without countability, then nothing like the present explication of symbolic pattern can suffice to witness all instances of truth-ground inclusion. At least, it cannot suffice if the patterns are even merely supposed to be the size of arbitrary sets. For if the set of elementary propositions is uncountable, then the relation of truth-ground inclusion is not Σ_1 definable in set theory.[45]

In surviving pre-*Tractatus* notebooks, Wittgenstein sometimes envisages that the totality of names be generated one after the other by a formal procedure. Perhaps if names must somehow be constructed by people then there could not be uncountably many of them. However, references to formal generability of the names disappear in the *Tractatus* itself. Wittgenstein may have supposed that idealism could coincide with realism only if the character of the class of names is not constrained in advance of analysis. It's not clear that the textual evidence suffices to determine whether countability of the names would count as such a constraint.

[45] See [[ref]] ?.

References

Dreben, B. and Floyd, J. (1991). Tautology: how not to use a word. *Synthese*, 87(1):23–49.

Frege, G. (1879). *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*. Louis Nebert, Halle.

Frege, G. (1884). *Die Grundlagen der Arithmetik: eine logisch-mathematische Untersuchung über den Begriff der Zahl*. W. Koebner, Breslau.

Frege, G. (1892). über sinn und bedeutung. *Zeitschrift für Philosophie und philosophische Kritik*, 100:25–50.

Frege, G. (1906). Über die grundlagen der geometrie (second series). *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 15:293–309; 377–403; 423–430.

Frege, G. (1918). Die Verneinung. Eine logische Untersuchung. *Beiträge zur Philosophie des deutschen Idealismus*, I:143–157.

Frege, G. (1979). *Posthumous Writings*. Blackwell.

Goldfarb, W. (2001). Wittgenstein's understanding of Frege: the pre-Tractarian evidence. In Reck, E., editor, *From Frege to Wittgenstein*. Oxford University Press.

Hart, W. D. (1971). The whole sense of the *Tractatus*. *The Journal of Philosophy*, 68(9):273–288.

Hilbert, D. (1912). *Grundzüge einer Allgemeinen Theorie der Linearen Integralgleichungen*. Druck und Verlag von B. G. Teubner.

Kremer, M. (1992). The multiplicity of general propositions. *Nous*, 26(4):409–426.

Martin-Löf, P. (1996). On the meanings of the logical constants and the justifications of the logical laws. *Nordic Journal of Formal Logic*, 1(1):11–60.

Proops, I. (1997). Wittgenstein on logical assertion. *Philosophical Topics*, 25(2):121–144.

Proops, I. (2002). The *Tractatus* on inference and entailment. In Reck, E., editor, *From Frege to Wittgenstein*, pages 283–307. Oxford University Press.

Ramsey, F. P. (1923). Critical notice of the *Tractatus*. *Mind*, 32(128):465–478.

Ricketts, T. (1985). Frege, the *Tractatus*, and the logocentric predicament. *Nous*, pages 3–15.

Rogers, B. and Wehmeier, K. (2012). Tractarian first-order logic: identity and the N-operator. *Review of Symbolic Logic*, 5(4):538–573.

Russell, B. (1903). *The Principles of Mathematics*. Allen and Unwin, London.

Shieh, S. (Forthcoming). In what sense does logic involve necessity? *Philosophical Topics*.

Sullivan, P. (2001). A version of the picture theory. http://www.philosophy.stir.ac.uk/staff/p-sullivan/documents/PIC-THRY.pdf.

Wittgenstein, L. (1922). *Tractatus Logico-Philosophicus*. Kegan Paul.

Wittgenstein, L. (1961). *Tractatus Logico-Philosophicus*. Routledge & Kegan Paul.

Wittgenstein, L. (1979). *Notebooks 1914-1916*. Blackwell, Oxford.

Zalabardo, J. (2010). The *Tractatus* on logical consequence. *European Journal of Philosophy*, 18:425–442.